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DYNAMICS AND CONTROL
OF MULTI-LINK ROBOT MANIPULATORS
WITH JOINT FLEXIBILITY

by

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Dynamics and Control of
Multi-Link Robot Manipulators With Joint Flexibility

by

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Lieutenant Commander, Taiwan Navy
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requirements for the degree of

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ABSTRACT

This thesis presents the dynamic modeling and its application to the multi-link robot manipulators with joint flexibility. The twist angles of springs are utilized to model the joint flexibility. A direct dynamic model is derived by the Lagrangian method and the Newton-Euler algorithm is used for deriving the inverse dynamics. A motion controller is designed by the computed torque method based on the inverse dynamics. A recursive estimator is included in the control design to estimate the required feedback signals which are not directly measurable.

C.1

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I. INTRODUCTION

The problem of joint flexibility has been recognized in most industrial robot designs. Joint flexibility typically results from the torsional flexibility of the gear mechanism and drive shafts. The effect of joint flexibility is negligible for low-speed and light payload operations and the rigid-body controllers are able to successfully control the system. The rigid-body controllers refer to controllers that are designed based on a rigid-body dynamic model. As the speed and payload increase, the inertia forces and gravity forces increase such that the flexibility effect becomes significant. The flexibility may not be tolerated by the rigid-body controller and may cause system instability. The improvement of the dynamic behavior requires that the joint flexibility effects be incorporated into the controller design for industrial robots. A dynamic model of industrial robots with joint flexibility is therefore needed.

The Lagrangian method was applied to rigid-body manipulators to obtain the direct dynamic model where the kinematics was described by a set of homogeneous transformation matrices [Ref. 1: pp. 158-172]. The recursive Newton-Euler method was used for solution of the inverse dynamic response to compute the required torques for rigid-body manipulators [Ref. 2: pp. 168-172]. Several nonlinear control techniques, namely feedback linearization, integral manifold and composite control [Ref. 3] were applied to design controllers for a single-link manipulator with joint elasticity. A position control was developed for a two-degree-of-freedom manipulator where the effects of drive-train compliance and

actuator dynamics were included [Ref. 4]. The control law was derived from the concepts of "computed torque" where the inverse dynamics were utilized to determine the required torques as the functions of position, velocity, acceleration, jerk and jerk rate errors. With the same control scheme, the computer simulation and dynamic analysis of a single-joint robot was conducted [Ref. 5]. The third joint of the PUMA 560 robot was selected as an example and a comparison between the performances of rigid-body controller and flexible-body controller was performed.

In this thesis, both the Lagrangian dynamics [Ref. 1] and the Newton-Euler algorithm [Ref. 2] will be applied to multi-link robot manipulators with joint flexibility. The former is used to derive the direct dynamics for the plant, computing the joint motion provided that the torque inputs are given. The latter is used to develop the inverse dynamics which returns the required torques assuming the joint motion is known. This thesis also uses the method of computed torque [Ref. 4] to design a control law for multi-link robot manipulators with joint flexibility. This control approach presents a method of decoupling the arm motion when drive-train compliance and actuator dynamics are considered. It is assumed that the angular positions and velocities of both links and motors are measurable from shaft encoders and tachometers. Since the second and third derivatives of link positions (i.e., accelerations and jerks) are required in the control design and not directly measurable, an estimator is designed in a recursive manner to obtain the second and third derivatives of link positions.

To present the frame work of this thesis, Figure 1 shows an error-driven feedback control system which includes the inverse dynamics (controller), direct dynamics (plant) and the estimator. In this figure, the system is designed to follow a reference input and the system error is generated by the difference between the desired reference input and the feedback. The Newton-Euler method is used to design the controller since its recursive formulation allows for the efficient computation of the inverse dynamics and therefore possibly for on-line control. The Lagrangian method derives the direct dynamics and models the plant for computer simulation because of its systematic nature which is convenient for programming.

The objective of this thesis is to formulate the direct and inverse dynamics of multi-link robot manipulators with joint flexibility. In addition, a computed-torque control is designed to show an application of the inverse dynamics. Notations and preliminaries will be presented in Chapter II. The direct dynamics and the inverse dynamics will be discussed in Chapter III and Chapter IV, respectively. Finally, the controller and estimator design procedures will be developed in Chapter V.

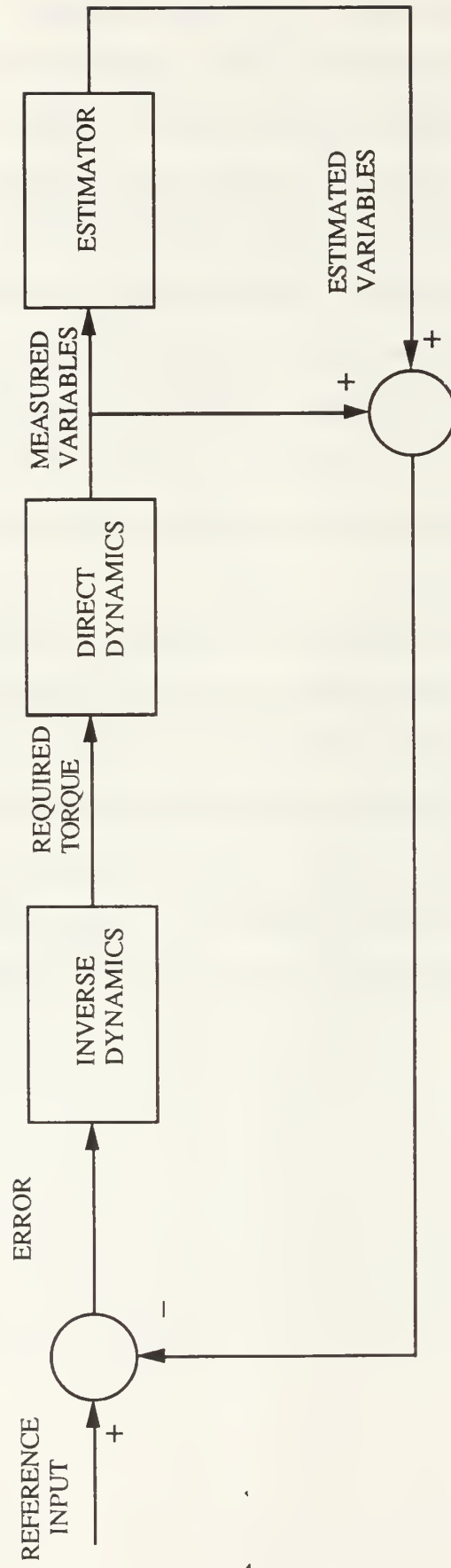


Figure 1. Error-Driven Feedback Control System

II. NOTATION AND PRELIMINARIES

This section defines the notations used in the thesis and briefly discusses some of the basic fundamentals which will be used to develop the model.

A. NOTATION AND PRELIMINARIES FOR THE DIRECT DYNAMICS

a_i : one of the four Hartenberg-Denavit parameters representing the distance between two adjacent frames $i-1$ and i in x direction.

α_i : one of the four Hartenberg-Denavit parameters representing the rotation angle about the x axis.

d_i : one of the four Hartenberg-Denavit parameters representing the distance between links in z direction.

θ_i : one of the four Hartenberg-Denavit parameters representing the rotation angle of links about the z axis.

q_i : joint variable of link i . For prismatic joints i , q_i is d_i , for revolute joint i , q_i is θ_i .

q_{mi} : the rotation angle of rotor i about the z axis.

G_i : gear ratio, i.e.,

$$G_i = \frac{\text{rotation angle of } i\text{-th motor shaft}}{\text{rotation angle of } i\text{-th link}} \quad (2.1)$$

A_i : the Hartenberg-Denavit matrix for link i , i.e.,

$$\mathbf{A}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_i \cos q_i & \cos q_i & -\sin q_i \cos \alpha_i & \sin q_i \sin \alpha_i \\ a_i \sin q_i & \sin q_i & \cos q_i \cos \alpha_i & -\cos q_i \sin \alpha_i \\ d_i & 0 & \sin \alpha_i & \cos \alpha_i \end{bmatrix} \quad (2.2)$$

$\mathbf{A}_{mi}$: homogeneous (4×4) coordinate transformation matrix of motor i ,

$$\text{i.e.,} \quad \mathbf{A}_{mi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos q_{mi} & -\sin q_{mi} & 0 \\ 0 & \sin q_{mi} & \cos q_{mi} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.3)$$

where

$$q_{mi} = G_i q_i - \delta_i \quad (2.4)$$

$\mathbf{W}_i$: homogeneous (4×4) coordinate transformation matrix from the i -th coordinate frame to the base coordinate frame.

$\mathbf{W}_i^j$: homogeneous (4×4) coordinate transformation matrix from the i -th coordinate frame to the j -th coordinate frame.

m_i : mass of the i -th link.

m_{mi} : mass of the i -th motor.

$\mathbf{r}_i^i$: local position vector of the center of mass of link i decomposed in the body-fixed coordinate.

$\mathbf{r}_i$: absolute position vector of the center of mass of link i decomposed in the base coordinate.

$\mathbf{r}_{mi}$: absolute position vector of the center of mass of motor i decomposed in the base coordinate.

$\mathbf{r}_{mi}^i$: local position vector of center of mass of motor i decomposed in the motor body- fixed coordinate.

$\mathbf{J}_{mi}$: the pseudo inertia matrix of the i -th motor

$$\mathbf{J}_{mi} = \int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{I_{mizz}}{2} & 0 & 0 \\ 0 & 0 & \frac{I_{mizz}}{2} & 0 \\ 0 & 0 & 0 & -\frac{I_{mizz}}{2} \end{bmatrix} \quad (2.5)$$

where I_{mizz} is the moment of inertia of motor i in the local z axis.

Note: The elements in the first row and the first column are zero since the mass m_{mi} of the motor i is considered to be included in the link i .

$\mathbf{J}_i$: the pseudo inertia matrix of the i -th link, where

$$\mathbf{J}_i = \int \mathbf{r}_i^i \mathbf{r}_i^{iT} dm$$

$$= \begin{bmatrix} m_i & m_i \bar{x}_i & m_i \bar{y}_i & m_i \bar{z}_i \\ m_i \bar{x}_i & \frac{-I_{ixx} + I_{iyy} + I_{izz}}{2} & I_{ixy} & I_{ixz} \\ m_i \bar{y}_i & I_{ixy} & \frac{I_{ixx} - I_{iyy} + I_{izz}}{2} & I_{iyz} \\ m_i \bar{z}_i & I_{ixz} & I_{iyz} & \frac{I_{ixx} + I_{iyy} - I_{izz}}{2} \end{bmatrix} \quad (2.6)$$

where $m_i \bar{x}_i$, $m_i \bar{y}_i$, $m_i \bar{z}_i$ are the first moments of link i , I_{ixx} , I_{iyy} , I_{izz} are the moments of the inertia of link i in the principal axes and I_{ixy} , I_{ixz} , I_{iyz} are the products of inertia of link i .

B. NOTATION AND PRELIMINARIES FOR INVERSE DYNAMICS

$\delta_i : G_i q_i - q_{mi}$.

ω_i : absolute angular velocity of link i .

$\dot{\omega}_i$: absolute angular acceleration of link i .

$\ddot{\omega}_i$: change rate of $\dot{\omega}_i$.

$\ddot{\ddot{\omega}}_i$: change rate of $\ddot{\omega}_i$.

Ω_i : absolute angular velocity of link i in a matrix form.

ω_{i-1} : local relative angular velocity of link i decomposed in coordinate $i-1$, i.e., $[0 \ 0 \ \dot{q}_i]^T$

$\dot{\omega}_{i-1}$: local relative angular acceleration of link i decomposed in coordinate $i-1$, i.e., $[0 \ 0 \ \ddot{q}_i]^T$

- $\ddot{\omega}_i$: local relative angular jerk of link i decomposed in coordinate $i-1$,
i.e., $[0 \ 0 \ \ddot{q}_i]^T$
- $\mathbf{a}_i$: linear acceleration of link i .
- ω_{mi} : absolute angular velocity of motor i decomposed in coordinate $i-1$.
- $\dot{\omega}_{mi}$: change rate of ω_{mi} .
- $\mathbf{a}_{ci}$: linear acceleration of link i at the center of mass.
- $\mathbf{F}_i$: resultant force exerted on link i at the center of mass.
- $\mathbf{f}_i$: force exerted on link i by link $i-1$.
- $\mathbf{N}_i$: resultant moment exerted on link i about the center of mass.
- $\mathbf{n}_i$: moment exerted on link i by link $i-1$.
- $\mathbf{n}_{mi}$: moment exerted on motor i decomposed in coordinate i .
- $\mathbf{T}_i$: torque generated from the electromagnetic field of motor i in the local z axis.
- $\mathbf{R}_i^j$: rotational transformation (3×3) matrix from the j -th coordinate frame to the i -th coordinate.
- K_{ij} : control gain.
- k_i : spring stiffness.
- $\mathbf{k}$: unit vector in the local z axis.
- $\mathbf{I}_i$: inertia tensor of link i , i.e.,

$$\mathbf{I}_i = \begin{bmatrix} I_{ixx} & I_{ixy} & I_{ixz} \\ I_{ixy} & I_{iyy} & I_{iyz} \\ I_{ixz} & I_{iyz} & I_{izz} \end{bmatrix} \quad (2.7)$$

I_{mi} : moments of inertia of motor i , i.e.,

$$I_{mi} = \begin{bmatrix} I_{mixx} & 0 & 0 \\ 0 & I_{miyy} & 0 \\ 0 & 0 & I_{mizz} \end{bmatrix} \quad (2.8)$$

III. DIRECT DYNAMICS

The Lagrangian dynamics approach is used to derive direct dynamics for the equation of motion of the plant in terms of the generalized coordinates. The Lagrangian dynamic equations are given as :

$$\frac{d}{dt} \left(\frac{\partial(KE)}{\partial \dot{\xi}_j} \right) - \frac{\partial(KE)}{\partial \xi_j} + \frac{\partial(PE)}{\partial \xi_j} = P_j \quad (j=1, 2, \dots, n) \quad (3.1)$$

where ξ_j is the generalized coordinate, n is the number of degrees of freedom and P_j denotes the generalized force.

This thesis considers a plant of a six-link manipulator with revolute joints actuated by DC-motors. Each motor is connected through a transmission shaft (Figure 2) to a rigid link and a spring with a constant stiffness is modeled to be a major source of joint flexibility. The twist angle of the spring δ_i is defined as $\delta_i = G_i q_i - q_{mi}$ where G_i is the gear ratio. Since additional degrees of freedom are introduced by the twisted spring, the generalized coordinates are defined as:

$$\xi = [q_1, q_2, \dots, q_6, \delta_1, \delta_2, \dots, \delta_6]^T$$

The generalized force P_j is obtained from the concept of virtual work.

Since

$$\delta W = \sum_{i=1}^6 T_i \delta q_{mi} = \sum_{i=1}^6 T_i (G_i \delta q_i - \delta \delta_i)$$

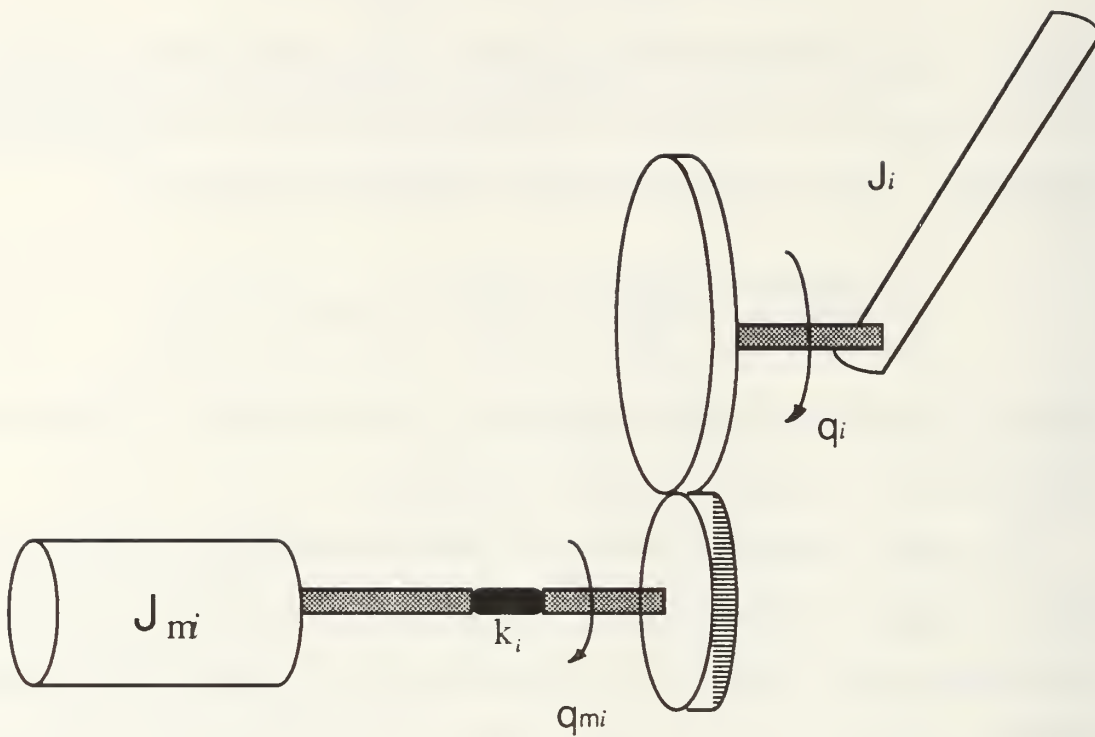


FIGURE 2. Single-Link Manipulator With Joint Flexibility

therefore, $T_i G_i$ is the generalized force corresponding to the generalized coordinate q_i , and $-T_i$ is the generalized force corresponding to the generalized coordinate δ_i .

The total kinetic energy of the system consists of the individual kinetic energies of the links and motors and the total potential energy is composed of the strain energy due to joint elasticity and the potential energy due to gravity. The kinetic energy KE and the potential energy PE are derived as follows: Given a point specified by a local position vector $\mathbf{r}_i^i$ (Figure 3), its position and velocity with respect to base coordinates are given as

$$\mathbf{r}_i = \mathbf{W}_i \mathbf{r}_i^i \quad (3.2)$$

and

$$\dot{\mathbf{r}}_i = \dot{\mathbf{W}}_i \mathbf{r}_i^i \quad (3.3)$$

The kinetic energy of link i is

$$(\text{KE})_{\text{LK}i} = \frac{1}{2} \text{tr} \int \dot{\mathbf{r}}_i \dot{\mathbf{r}}_i^T \text{dm} \quad (3.4)$$

For six-link robot manipulators, the total kinetic energy becomes

$$\begin{aligned} (\text{KE})_{\text{LK}} &= \sum_{i=1}^6 (\text{KE})_{\text{LK}i} \\ &= \sum_{i=1}^6 \frac{1}{2} \text{tr} \int \dot{\mathbf{r}}_i \dot{\mathbf{r}}_i^T \text{dm} \end{aligned} \quad (3.5)$$

Substituting Equation (3.3) into Equation (3.5) gives

$$(\text{KE})_{\text{LK}} = \sum_{i=1}^6 \frac{1}{2} \text{tr} \left[\dot{\mathbf{W}}_i \left(\int \mathbf{r}_i^i \mathbf{r}_i^{iT} \text{dm} \right) \dot{\mathbf{W}}_i \right] \quad (3.6)$$

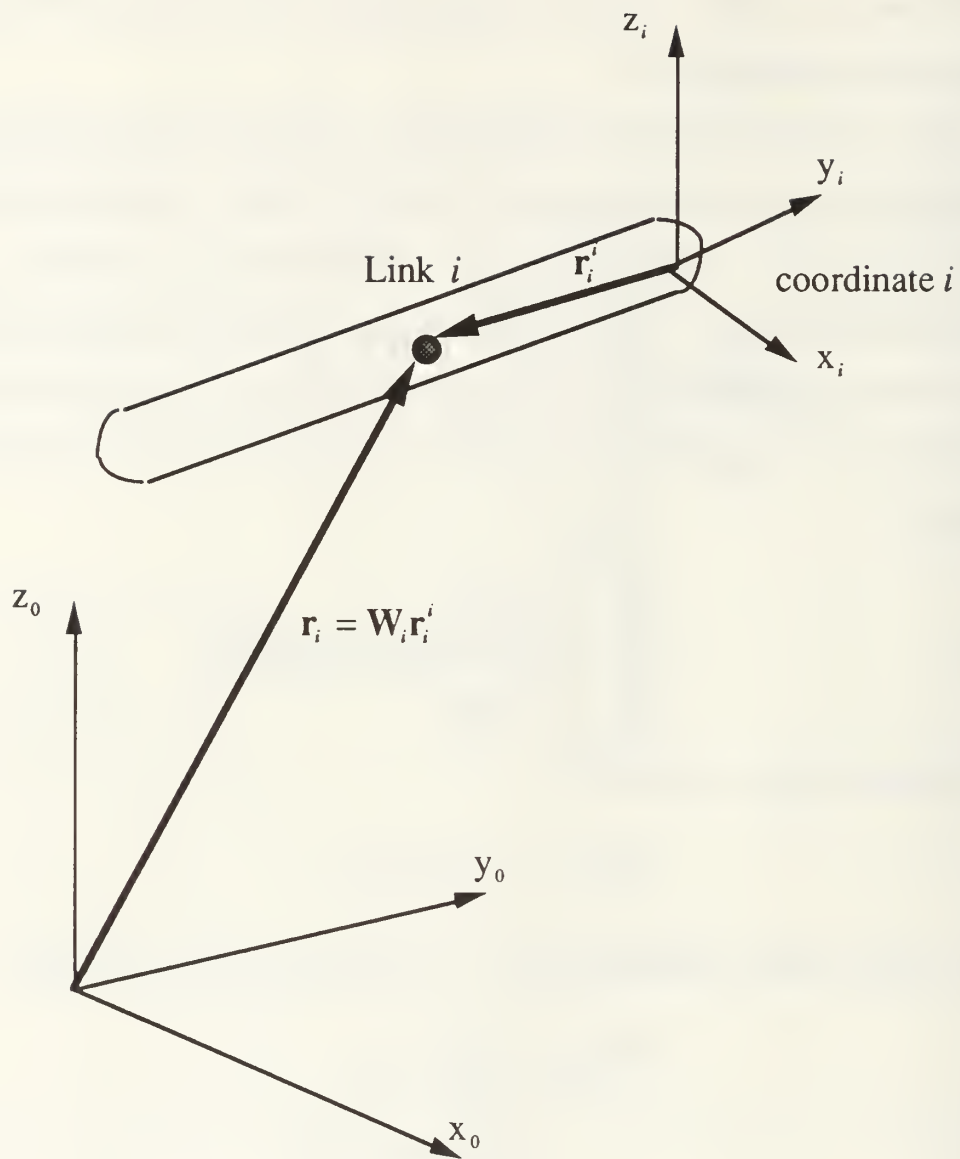


Figure 3. Position Vector of The Center of Mass of Link i

The integral in Equation (3.6) is known as the pseudo inertia matrix J_i and was given in section II. Equation (3.6) thus becomes

$$(\text{KE})_{\text{LK}} = \sum_{i=1}^6 \frac{1}{2} \text{tr} [\dot{\mathbf{W}}_i \mathbf{J}_i \dot{\mathbf{W}}_i] \quad (3.7)$$

Similarly, the kinetic energy of the motors is

$$(\text{KE})_{\text{m}} = \sum_{i=1}^6 \frac{1}{2} \text{tr} \int \dot{\mathbf{r}}_{mi} \dot{\mathbf{r}}_{mi}^T dm \quad (3.8)$$

where

$$\mathbf{r}_{mi} = \mathbf{W}_{i-1} \mathbf{A}_{mi} \mathbf{r}_{mi}^i \quad (3.9)$$

The time derivative of $\mathbf{r}_{mi}$ is obtained as

$$\dot{\mathbf{r}}_{mi} = \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{r}_{mi}^i + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{r}_{mi}^i \quad (3.10)$$

and

$$\mathbf{A}_{mi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos q_{mi} & -\sin q_{mi} & 0 \\ 0 & \sin q_{mi} & \cos q_{mi} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.11)$$

Substituting Equation (3.10) into Equation (3.8) gives

$$\begin{aligned} (\text{KE})_{\text{m}} &= \sum_{i=1}^6 \frac{1}{2} \text{tr} \left[\left(\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{r}_{mi}^i + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{r}_{mi}^i \right) \right. \\ &\quad \left. \cdot \left(\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{r}_{mi}^i + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{r}_{mi}^i \right)^T \right] dm \\ &= \frac{1}{2} \text{tr} \left[\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \left(\int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \right) \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right. \\ &\quad \left. + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \left(\int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \right) \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \\ &\quad \left. + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \left(\int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \right) \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \\ &\quad \left. + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \left(\int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \right) \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right] \end{aligned}$$

$$\begin{aligned}
& + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \left(\int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \right) \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \\
& + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \left(\int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \right) \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \Big] \quad (3.12)
\end{aligned}$$

Since

$$\text{tr} \left[\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \left(\int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \right) \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right] = \text{tr} \left[\mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \left(\int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \right) \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right] \quad (3.13)$$

Equation (3.12) becomes

$$\begin{aligned}
(\text{KE})_m &= \sum_{i=1}^6 \frac{1}{2} \text{tr} \left[\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right. \\
&\quad + 2 \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \\
&\quad \left. + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right] \quad (3.14)
\end{aligned}$$

where

$$\mathbf{J}_{mi} = \int \mathbf{r}_{mi}^i \mathbf{r}_{mi}^{iT} dm \quad (3.15)$$

and the total kinetic energy of the system can be found as

$$(\text{KE}) = (\text{KE})_{\text{LK}} + (\text{KE})_m \quad (3.16)$$

The potential energies of links and motors are obtained as

$$\begin{aligned}
(\text{PE})_{\text{LK}} &= - \sum_{i=1}^6 m_i \mathbf{r}_{ci}^T \mathbf{g} \\
&= - \sum_{i=1}^6 m \mathbf{r}_{ci}^{iT} \mathbf{W}_i^T \mathbf{g} \quad (3.17)
\end{aligned}$$

and

$$\begin{aligned}
(\text{PE})_m &= - \sum_{i=1}^6 m_{mi} \mathbf{r}_{cmi}^T \mathbf{g} \\
&= - \sum_{i=1}^6 m_{mi} \left(\mathbf{W}_{i-1} \mathbf{A}_{mi} \mathbf{r}_{cmi}^i \right)^T \mathbf{g} \\
&= - \sum_{i=1}^6 m_{mi} \mathbf{r}_{cmi}^{iT} \mathbf{A}_{mi}^T \mathbf{W}_{i-1}^T \mathbf{g} \quad (3.18)
\end{aligned}$$

where

$$\mathbf{g} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -9.8 \end{Bmatrix} \quad \left(\frac{\text{m}}{\text{sec}^2} \right)$$

The center of mass of link i is described by a local vector $\mathbf{r}_{ci}^i$ and the center of mass of motor i is described by a local vector $\mathbf{r}_{cmi}^i$. Their position transformations from local coordinate to base coordinate are defined as $\mathbf{r}_{ci} = \mathbf{W}_i \mathbf{r}_{ci}^i$ and $\mathbf{r}_{cmi} = \mathbf{W}_{i-1} \mathbf{A}_{mi} \mathbf{r}_{cmi}^i$, respectively.

The potential energy due to strain is

$$(\text{PE})_s = \frac{1}{2} \sum_{i=1}^6 k_i \delta_i^2 \quad (3.19)$$

Therefore, the total potential energy of the system is obtained by combining Equations (3.17), (3.18) and (3.19) :

$$(\text{PE}) = - \sum_{i=1}^6 m_i \mathbf{r}_{ci}^{i\top} \mathbf{W}_i^\top \mathbf{g} - \sum_{i=1}^6 m_{mi} \mathbf{r}_{cmi}^{i\top} \mathbf{A}_{mi}^\top \mathbf{W}_{i-1}^\top \mathbf{g} + \frac{1}{2} \sum_{i=1}^6 k_i \delta_i^2 \quad (3.20)$$

Since generalized coordinate q_i is for large motion and generalized coordinate δ_i is for small motion, two sets of equation of motion will be obtained according to Equation (3.1). For the large motion, the general form of dynamic equation is described as

$$\sum_{i=j}^6 \text{tr}(\mathbf{W}_{i,j} \mathbf{J}_i \ddot{\mathbf{W}}_i^\top) + \text{tr}(\ddot{\mathbf{X}}_j \mathbf{J}_{mj} \mathbf{A}_{mj,j}^\top \mathbf{W}_{j-1}^\top) + \sum_{i=j+1}^6 \text{tr}(\ddot{\mathbf{X}}_j \mathbf{J}_{mj} \mathbf{A}_{mj}^\top \mathbf{W}_{i-1,j}^\top)$$

$$-\sum_{i=j}^6 m_i \mathbf{r}_{ci}^{i^T} \mathbf{W}_{i,j}^T \mathbf{g} - m_{mj} \mathbf{r}_{cmj}^{j^T} \mathbf{A}_{mj,j}^T \mathbf{W}_{j-1}^T \mathbf{g} - \sum_{i=j+1}^6 m_{mi} \mathbf{r}_{cmi}^{i^T} \mathbf{A}_{mi}^T \mathbf{W}_{i-1,j}^T \mathbf{g} = \mathbf{G}_j \mathbf{T}_j \quad (3.21)$$

where

$$\mathbf{X}_j = \mathbf{W}_{j-1} \mathbf{A}_{mj} \quad (3.22)$$

$$\dot{\mathbf{X}}_j = \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} + \mathbf{W}_{j-1} \dot{\mathbf{A}}_{mj} \quad (3.23)$$

$$\ddot{\mathbf{X}}_j = \ddot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} + 2\dot{\mathbf{W}}_{j-1} \dot{\mathbf{A}}_{mj} \mathbf{W}_{j-1} \ddot{\mathbf{A}}_{mj} \quad (3.24)$$

The dynamic equation for the small motion is described as

$$-\text{tr}(\ddot{\mathbf{X}}_j \mathbf{J}_{mj} \mathbf{A}_{mj,j}^T \mathbf{W}_{i-1}^T) + m_{mj} \mathbf{r}_{mj}^{j^T} \dot{\mathbf{A}}_{mj,j}^T \mathbf{W}_{j-1}^T \mathbf{g} + k_j \delta_j = -\mathbf{T}_j \quad (3.25)$$

The second time derivative of the transformation matrix $\ddot{\mathbf{W}}_i$ can be derived as

$$\ddot{\mathbf{W}}_i = \sum_{k=1}^i \mathbf{W}_{i,k} \ddot{\mathbf{q}}_k + \sum_{k=1}^i \sum_{m=1}^i \mathbf{W}_{i,km} \dot{\mathbf{q}}_k \dot{\mathbf{q}}_m \quad (3.26)$$

where $\sum_{k=1}^i \sum_{m=1}^i \mathbf{W}_{i,km} \dot{\mathbf{q}}_k \dot{\mathbf{q}}_m$ is defined as the residual term that includes Coriolis and centripetal accelerations. A recursive form (see Appendix A) is used to compute the residual term to avoid tedious computation. The detailed procedure of direct dynamic equations is derived in Appendix B.

For the special case of a rigid body plant, the stiffness of the spring approaches infinity. Therefore, the elastic displacement is equal to zero, and Equation (3.21) for the rigid body case reduces to :

$$\sum_{i=j}^6 \text{tr}(\mathbf{W}_{i,j} \mathbf{J}_i \ddot{\mathbf{W}}_i^T) - \sum_{i=j}^6 m_i \mathbf{r}_i^{i^T} \mathbf{W}_{i,j}^T \mathbf{g} = \mathbf{T}_j \quad (3.27)$$

In this case, the Equation (3.25) presenting the small motion can be ignored directly.

IV. INVERSE DYNAMICS

The inverse dynamics is to predict either the required driving force or torque for a mechanical system of which the motion (i.e., the joint positions, velocities, accelerations and jerks in this thesis) is known. The inverse dynamics of robot manipulators with joint flexibility is developed in this section by the Newton-Euler method. The controller and the estimator are designed based on the inverse dynamics for that is computationally efficient and has potential for on-line control.

Figure 4(a) presents a free body diagram of the link i where $\mathbf{f}_i$ and $\mathbf{n}_i$ are force and torque exerted by link $i-1$ decomposed in coordinate i and $\mathbf{f}_{i+1}$ and $\mathbf{n}_{i+1}$ are force and torque exerted by link $i+1$ decomposed in coordinate $i+1$. This system of forces and moments is equivalent to that on Figure 4(b) where $\mathbf{F}_i$ and $\mathbf{N}_i$ are the resultant force and torque acting on the center of mass. Therefore, the force and moment balance equations are obtained as

$$\mathbf{F}_i = \mathbf{f}_i - \mathbf{R}_i^{i+1} \mathbf{f}_{i+1} \quad (4.1)$$

$$\mathbf{N}_i = \mathbf{n}_i - \mathbf{R}_i^{i+1} \mathbf{n}_{i+1} - \mathbf{S} \mathbf{l}_i \times \mathbf{f}_i - \mathbf{C} \mathbf{l}_i \times \mathbf{F}_i \quad (4.2)$$

The Newton-Euler equations can then be written as

$$\mathbf{F}_i = m_i \mathbf{a}_{ci} \quad (4.3)$$

$$\mathbf{N}_i = \mathbf{I}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\Omega}_i (\mathbf{I}_i \boldsymbol{\omega}_i) \quad (4.4)$$

By taking balance of moments in the local z direction on the free body diagram of the motor i (the bottom figure of Figure 5), the torque equation can be expressed as

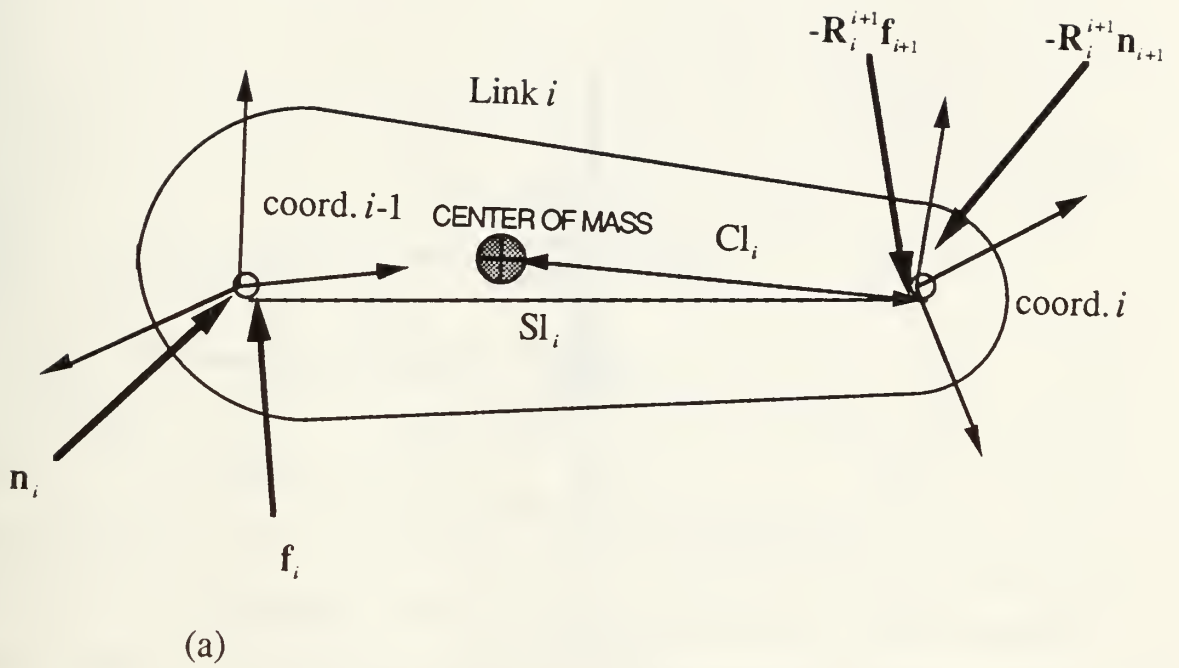


Figure 4(a). Free-Body Diagram of Link i

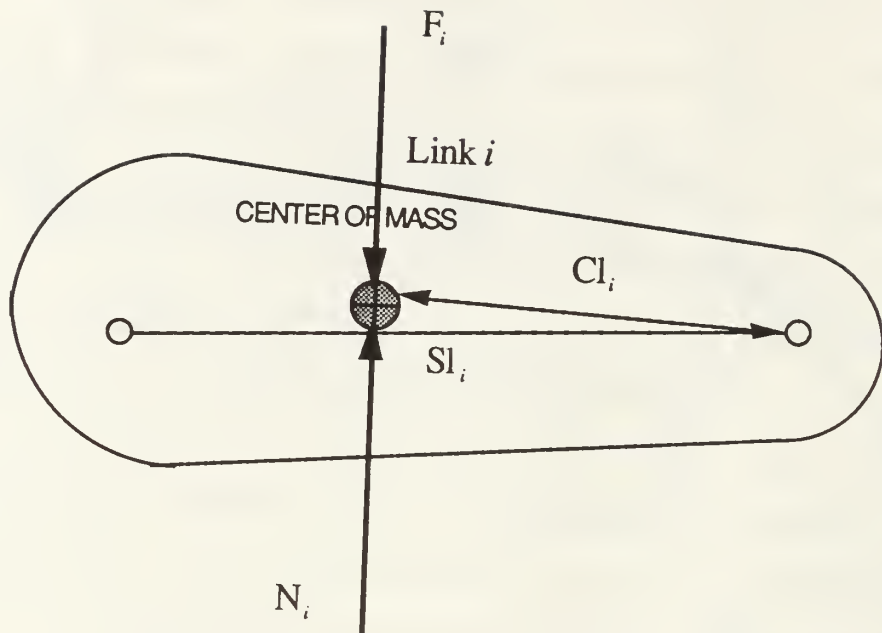


Figure 4(b). Resultant Force and Torque of Link i

$$T_i = (\mathbf{R}_{i-1}^i \mathbf{n}_i) \cdot \mathbf{k} + (\mathbf{I}_{mi} \dot{\omega}_{mi} + \Omega_{i-1} \mathbf{I}_{mi} \omega_{mi}) \cdot \mathbf{k} \quad (4.5)$$

where

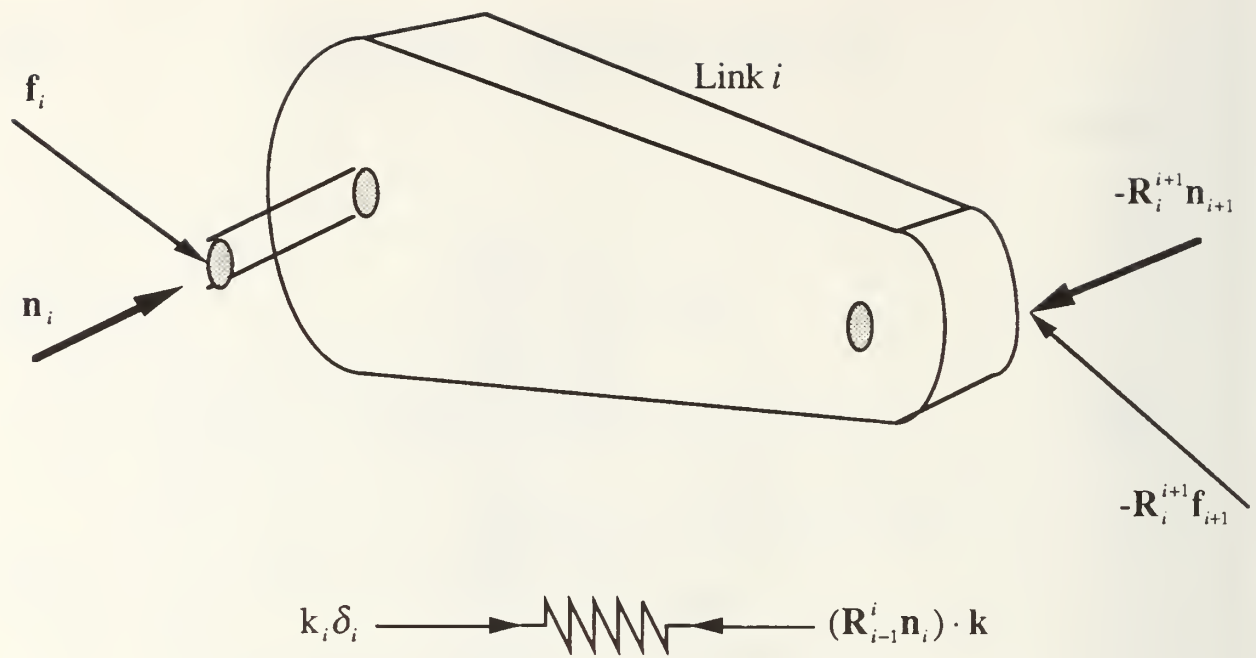
$$\omega_{mi} = \omega_{i-1} + \begin{Bmatrix} 0 \\ 0 \\ \dot{q}_{mi} \end{Bmatrix} \quad (4.6)$$

and

$$\dot{\omega}_{mi} = \dot{\omega}_{i-1} + \begin{Bmatrix} 0 \\ 0 \\ \ddot{q}_{mi} \end{Bmatrix} \quad (4.7)$$

From Equations (4.5), (4.6) and (4.7), the required torque T_i can be obtained if the kinematics of the manipulator is given. The recursive inverse dynamics of rigid-body systems has been available [Ref. 6] where the gravity has been incorporated into the kinematics. For flexible-joint systems, the motor variables are not identical to the link variables and $\dot{q}_{mi}$ and $\ddot{q}_{mi}$ will be related to link variables in this research.

Figure 5 shows three free-body diagrams, i.e., free-body diagram of link i , the i -th spring and motor i , which illustrate the relations among the moment $\mathbf{n}_i$, the spring torque $k_i \delta_i$ and the torque T_i produced by the electromagnetic field of the motor. Since only the moments in local z axis are concerned, the moments acting on the both sides of the spring along z axis are equivalent in magnitude and opposite in directions. In other words, the component of the moment $\mathbf{n}_i$ in the local z axis acting on link i , i.e., $(\mathbf{R}_{i-1}^i \mathbf{n}_i) \cdot \mathbf{k}$, is equal to that acting on the spring in the local z axis. This relation can be given as



$$k_i \delta_i = k_i (G_i q_i - q_{m_i}) = (\mathbf{R}_{i-1}^i \mathbf{n}_i) \cdot \mathbf{k}$$

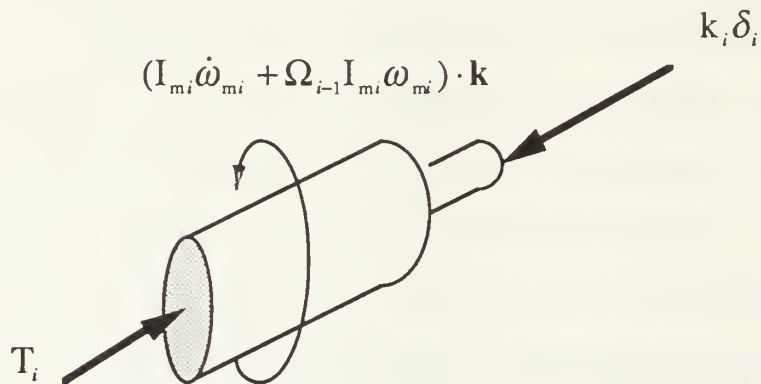


Figure 5. Free-Body diagram of a link, a spring and a motor

$$\mathbf{k}_i \delta_i = (\mathbf{R}_{i-1}^i \mathbf{n}_i) \cdot \mathbf{k} \quad (4.8)$$

or

$$\delta_i = \frac{1}{k_i} (\mathbf{R}_{i-1}^i \mathbf{n}_i) \cdot \mathbf{k} \quad (4.9)$$

As we defined δ_i in section II, q_{mi} can be described as

$$q_{mi} = G_i q_i - \delta_i \quad (4.10)$$

By substituting Equation (4.9) into Equation (4.10), the angular position of motor i , i.e., q_{mi} , can be expressed as a function of generalized coordinate q_i :

$$q_{mi} = G_i q_i - \frac{1}{k_i} (\mathbf{R}_{i-1}^i \mathbf{n}_i) \cdot \mathbf{k}$$

The above equation is equivalent to the following equation where the vectors are decomposed in the base coordinates.

$$q_{mi} = G_i q_i - \frac{1}{k_i} (\mathbf{R}_0^i \mathbf{n}_i) \cdot (\mathbf{R}_0^{i-1} \mathbf{k}) \quad (4.14)$$

As referring to Equation (4.7), the first and second time derivatives of angular position of motor i are required to compute the torque T_i . By applying $\dot{\mathbf{R}}_0^j = \mathbf{R}_0^j \Omega_j$, $\dot{q}_{mi}$ and $\ddot{q}_{mi}$ can be obtained as

$$\begin{aligned} \dot{q}_{mi} &= G_i \dot{q}_i - \frac{1}{k_i} \left[\mathbf{R}_0^i (\Omega_i \mathbf{n}_i + \dot{\mathbf{n}}_i) \cdot (\mathbf{R}_0^{i-1} \mathbf{k}) + (\mathbf{R}_0^i \mathbf{n}_i) \cdot (\mathbf{R}_0^{i-1} \Omega_{i-1} \mathbf{k}) \right] \quad (4.15) \\ \ddot{q}_{mi} &= G_i \ddot{q}_i - \frac{1}{k_i} \left\{ \mathbf{R}_0^i (\Omega_i \Omega_i \mathbf{n}_i + \dot{\Omega}_i \mathbf{n}_i + 2\Omega_i \dot{\mathbf{n}}_i + \ddot{\mathbf{n}}_i) \cdot (\mathbf{R}_0^{i-1} \mathbf{k}) \right. \\ &\quad \left. + 2\mathbf{R}_0^i (\Omega_i \mathbf{n}_i + \dot{\mathbf{n}}_i) \cdot (\mathbf{R}_0^{i-1} \Omega_{i-1} \mathbf{k}) + (\mathbf{R}_0^i \mathbf{n}_i) \cdot \left[\mathbf{R}_0^{i-1} (\dot{\Omega}_{i-1} \mathbf{k} + \Omega_{i-1} \Omega_{i-1} \mathbf{k}) \right] \right\} \end{aligned}$$

(4.16)

where

$$\mathbf{n}_i = \mathbf{N}_i + \mathbf{R}_i^{i+1} \mathbf{n}_{i+1} + \mathbf{S} \mathbf{l}_i \times \mathbf{f}_i + \mathbf{C} \mathbf{l}_i \times \mathbf{F}_i \quad (4.17)$$

$$\begin{aligned} \dot{\mathbf{n}}_i = & \Omega_i \mathbf{N}_i + \dot{\mathbf{N}}_i + \mathbf{R}_i^{i+1} (\Omega_{i+1} \mathbf{n}_{i+1} + \dot{\mathbf{n}}_{i+1}) + \Omega_i (\mathbf{S} \mathbf{l}_i \times \mathbf{f}_i) + \mathbf{S} \mathbf{l}_i \times \dot{\mathbf{f}}_i \\ & + \Omega_i (\mathbf{C} \mathbf{l}_i \times \mathbf{F}_i) + \mathbf{C} \mathbf{l}_i \times \dot{\mathbf{F}}_i - \Omega_i \mathbf{n}_i \end{aligned} \quad (4.18)$$

$$\begin{aligned} \ddot{\mathbf{n}}_i = & \Omega_i \Omega_i \mathbf{N}_i + 2\Omega_i \dot{\mathbf{N}}_i + \dot{\Omega}_i \mathbf{N}_i + \ddot{\mathbf{N}}_i + \mathbf{R}_i^{i+1} (\Omega_{i+1} \Omega_{i+1} \mathbf{n}_{i+1} \\ & + 2\Omega_{i+1} \dot{\mathbf{n}}_{i+1} + \dot{\Omega}_{i+1} \mathbf{n}_{i+1} + \ddot{\mathbf{n}}_{i+1}) + \Omega_i \Omega_i (\mathbf{S} \mathbf{l}_i \times \mathbf{f}_i) \\ & + 2\Omega_i (\mathbf{S} \mathbf{l}_i \times \dot{\mathbf{f}}_i) + \dot{\Omega}_i (\mathbf{S} \mathbf{l}_i \times \mathbf{f}_i) + \mathbf{S} \mathbf{l}_i \times \ddot{\mathbf{f}}_i + \Omega_i \Omega_i (\mathbf{C} \mathbf{l}_i \times \mathbf{F}_i) \\ & + 2\Omega_i (\mathbf{C} \mathbf{l}_i \times \dot{\mathbf{F}}_i) + \dot{\Omega}_i (\mathbf{C} \mathbf{l}_i \times \mathbf{F}_i) + \mathbf{C} \mathbf{l}_i \times \ddot{\mathbf{F}}_i - \Omega_i \Omega_i \mathbf{n}_i \\ & - 2\Omega_i \dot{\mathbf{n}}_i - \dot{\Omega}_i \mathbf{n}_i \end{aligned} \quad (4.19)$$

$$\mathbf{N}_i = \mathbf{I}_i \dot{\omega}_i + \Omega_i (\mathbf{I}_i \omega_i) \quad (4.20)$$

$$\dot{\mathbf{N}}_i = \mathbf{I}_i \ddot{\omega}_i + \dot{\Omega}_i \mathbf{I}_i \omega_i + \Omega_i \mathbf{I}_i \dot{\omega}_i \quad (4.21)$$

$$\ddot{\mathbf{N}}_i = \mathbf{I}_i \ddot{\omega}_i + \ddot{\Omega}_i \mathbf{I}_i \omega_i + 2\dot{\Omega}_i \mathbf{I}_i \dot{\omega}_i + \Omega_i \mathbf{I}_i \ddot{\omega}_i \quad (4.22)$$

$$\omega_i = \mathbf{R}_i^{i-1} (\omega_{i-1} + \omega_{l,i}) \quad (4.23)$$

$$\dot{\omega}_i = \mathbf{R}_i^{i-1} (\Omega_{i-1} \omega_{l,i} + \dot{\omega}_{i-1} + \dot{\omega}_{l,i}) \quad (4.24)$$

$$\begin{aligned} \ddot{\omega}_i = & \mathbf{R}_i^{i-1} (\Omega_{i-1} \Omega_{i-1} \omega_{l,i} + \Omega_{i-1} \dot{\omega}_{i-1} + 2\Omega_{i-1} \dot{\omega}_{l,i} + \dot{\Omega}_{i-1} \dot{\omega}_{l,i} + \ddot{\omega}_{i-1} \\ & + \ddot{\omega}_{l,i}) - \Omega_i \dot{\omega}_i \end{aligned} \quad (4.25)$$

$$\begin{aligned} \ddot{\omega}_i = & \mathbf{R}_i^{i-1} (\Omega_{i-1} \Omega_{i-1} \Omega_{i-1} \omega_{l,i} + \Omega_{i-1} \Omega_{i-1} \dot{\omega}_{i-1} + 3\Omega_{i-1} \Omega_{i-1} \dot{\omega}_{l,i} \\ & + 2\Omega_{i-1} \dot{\Omega}_{i-1} \omega_{l,i} + 2\Omega_{i-1} \ddot{\omega}_{i-1} + 3\Omega_{i-1} \ddot{\omega}_{l,i} + \dot{\Omega}_{i-1} \Omega_{i-1} \omega_{l,i} \\ & + \dot{\Omega}_{i-1} \dot{\omega}_{i-1} + 3\dot{\Omega}_{i-1} \omega_{l,i} + \ddot{\Omega}_{i-1} \omega_{l,i} + \ddot{\omega}_{i-1} + \ddot{\omega}_{l,i}) \\ & - (\Omega_i \Omega_i \dot{\omega}_i + 2\Omega_i \ddot{\omega}_i + \dot{\Omega}_i \dot{\omega}_i) \end{aligned} \quad (4.26)$$

The intermediate variables $\mathbf{f}_i$, $\mathbf{F}_i$, and their rates of change are described in Appendix C.

Since

$$\omega_{l,i} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\mathbf{q}}_i \end{Bmatrix} \quad (4.27)$$

differentiating Equation (4.27) with respect to time gives

$$\dot{\omega}_{l,i} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\mathbf{q}}_i \end{Bmatrix}, \quad \ddot{\omega}_{l,i} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\mathbf{q}}_i \end{Bmatrix}, \quad \dddot{\omega}_{l,i} = \begin{Bmatrix} 0 \\ 0 \\ \mathbf{q}_i^{IV} \end{Bmatrix} \quad (4.28)$$

Substituting Equations (4.27-28) into Equation (4.5) yields a fourth-order ordinary differential equation. The inverse dynamics can be simply describes as

$$\mathbf{T}_i = \mathbf{J}_i^* \mathbf{q}_i^{IV} + \mathbf{f}_i \quad (4.29)$$

where $\mathbf{f}_i$ is a nonlinear function of arm position, velocity, acceleration and jerk, the term $\mathbf{J}_i^*$ is a function of the moments of inertia of motors and links and spring constants. $\mathbf{T}_i$ is the required torque.

IV. CONTROLLER AND ESTIMATOR DESIGN

The purpose of designing an controller for the manipulator is to drive the tracking error to zero asymptotically. In this section, a method of computed torque [Ref. 4] is utilized as a control scheme to determine the required torques.

Define the error as

$$\varepsilon = q_{iD} - \hat{q}_i \quad (5.1)$$

where q_{iD} is the desired angular position of link i and $\hat{q}_i$ is the measured angular position of link i . The error dynamics for the fourth-order system (Equation (4.29)) is described as:

$$\varepsilon^{IV} + K_{i3}\ddot{\varepsilon} + K_{i2}\dot{\varepsilon} + K_{i1}\varepsilon = 0 \quad (5.2)$$

where the K 's are constants representing the feedback gains. By substituting q_i^{IV} in Equation (4.29) to the error Equation (5.2), the computed torque is then obtained as

$$T_i = J_{ical}^* q_{ical}^{IV} + f_{ical} \quad (5.3)$$

where J_{ical}^* is a calculated value of J_i^* and f_{ical} is the calculated value of f_i . q_{ical}^{IV} is the "calculated jerk rate" given by

$$q_{ical}^{IV} = q_{iD}^{IV} + k_{i3}(\ddot{q}_{iD} - \ddot{\hat{q}}_i) + k_{i2}(\dot{q}_{iD} - \dot{\hat{q}}_i) + k_{i1}(q_{iD} - \hat{q}_i) \quad (5.4)$$

where $\ddot{\hat{q}}_i, \dot{\hat{q}}_i, \hat{q}_i$ and $\hat{q}_i$ are measured variables. J_{ical}^* and f_{ical} can be computed from Equation (4.29) based on the complete knowledge of the plant dynamics and availability of measurements.

ITAE performance criteria [Ref. 7] for a fourth-order equation is used for the selection of the k values. The values are

$$\begin{aligned}
k_{i3} &= 2.1\omega_n \\
k_{i2} &= 3.4\omega_n^2 \\
k_{i1} &= 2.7\omega_n^3 \\
k_{i0} &= 1.0\omega_n^4
\end{aligned} \tag{5.5}$$

where ω_n is a selected value for the servo bandwidth.

For a rigid-body case, the stiffnesses of all springs approach infinity, and the δ_i approaches zero. Equation (5.3) is reduced to a second-order ordinary differential equation :

$$T_i = J_{ical}^* \ddot{q}_{ical} + f_{ical} \tag{5.6}$$

where

$$\ddot{q}_{ical} = \ddot{q}_{iD} + k_{i1}(\dot{q}_{iD} - \dot{\hat{q}}_i) + k_{i0}(q_{iD} - \hat{q}_i) \tag{5.7}$$

and the values of the feedback gains according to the ITAE are :

$$k_{i1} = 1.4\omega_n \tag{5.9}$$

$$k_{i0} = 1.0\omega_n^2 \tag{5.10}$$

The fourth-order system requires four measurements for each link to compute the driving torques. It was assumed that the angular positions and velocities of both links and motors are available from shaft encoders and tachometers. The second and third derivatives of link positions (i.e., accelerations and jerks) are not measurable. Equation (5.4) shows that the position, velocity, acceleration and jerk of link are required to compute the torques to drive the plant. An estimator in a recursive form is designed to estimate those variables that are not measurable (i.e., $\ddot{q}_i$ and $\ddot{\dot{q}}_i$) by using the measurements from those variables that are accessible (i.e., $q_i, \dot{q}_i, q_{mi}$ and $\dot{q}_{mi}$).

By substituting Equation (4.20) into Equation (4.17), it becomes

$$\mathbf{n}_i = \mathbf{I}_i \dot{\omega}_i + \Omega_i(\mathbf{I}_i \omega_i) + \mathbf{R}_i^{i+1} \mathbf{n}_{i+1} + \mathbf{S} \mathbf{I}_i \times \mathbf{f}_i + \mathbf{C} \mathbf{I}_i \times \mathbf{F}_i \tag{5.11}$$

then substituting Equation (4.24) into Equation (5.11) gives

$$\begin{aligned} \mathbf{n}_i = & \mathbf{I}_i \mathbf{R}_i^{i-1} (\Omega_{i-1} \omega_{l,i} + \dot{\omega}_{i-1} + \dot{\omega}_{l,i}) + \Omega_i (\mathbf{I}_i \omega_i) + \mathbf{R}_i^{i+1} \mathbf{n}_{i+1} \\ & + \mathbf{S} \mathbf{l}_i \times \mathbf{f}_i + \mathbf{C} \mathbf{l}_i \times \mathbf{F}_i \end{aligned} \quad (5.12)$$

Substituting Equation (5.12) into Equation (4.8) yields

$$\begin{aligned} k_i \delta_i = & \left\{ \mathbf{R}_{i-1}^i \left[\mathbf{I}_i \mathbf{R}_i^{i-1} (\Omega_{i-1} \omega_{l,i} + \dot{\omega}_{i-1} + \dot{\omega}_{l,i}) + \Omega_i (\mathbf{I}_i \omega_i) \right. \right. \\ & \left. \left. + \mathbf{R}_i^{i+1} \mathbf{n}_{i+1} + \mathbf{S} \mathbf{l}_i \times \mathbf{f}_i + \mathbf{C} \mathbf{l}_i \times \mathbf{F}_i \right] \right\} \cdot \mathbf{k} \end{aligned} \quad (5.13)$$

Equation (5.13) is rearranged as

$$\begin{aligned} (\mathbf{R}_{i-1}^i \mathbf{I}_i \mathbf{R}_i^{i-1} \dot{\omega}_{l,i}) \cdot \mathbf{k} = & k_i \delta_i - \left\{ \mathbf{R}_{i-1}^i \left[\mathbf{I}_i \mathbf{R}_i^{i-1} (\Omega_{i-1} \omega_{l,i} + \dot{\omega}_{i-1}) + \Omega_i (\mathbf{I}_i \omega_i) \right. \right. \\ & \left. \left. - \mathbf{R}_i^{i+1} \mathbf{n}_{i+1} + \mathbf{S} \mathbf{l}_i \times \mathbf{f}_i + \mathbf{C} \mathbf{l}_i \times \mathbf{F}_i \right] \right\} \cdot \mathbf{k} \end{aligned} \quad (5.14)$$

Recall Equation (4.28) where

$$\dot{\omega}_{l,i} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{q}_i \end{Bmatrix}$$

therefore Equation (5.14) can be rewritten as

$$\begin{aligned} \ddot{q}_i = & \frac{1}{(\mathbf{R}_{i-1}^i \mathbf{I}_i \mathbf{R}_i^{i-1})_{33}} \left(k_i \delta_i - \left\{ \mathbf{R}_{i-1}^i \left[\mathbf{I}_i \mathbf{R}_i^{i-1} (\Omega_{i-1} \omega_{l,i} + \dot{\omega}_{i-1}) + \Omega_i (\mathbf{I}_i \omega_i) \right. \right. \right. \\ & \left. \left. - \mathbf{R}_i^{i+1} \mathbf{n}_{i+1} + \mathbf{S} \mathbf{l}_i \times \mathbf{f}_i + \mathbf{C} \mathbf{l}_i \times \mathbf{F}_i \right] \right\} \cdot \mathbf{k} \right) \end{aligned} \quad (5.15)$$

where $(\mathbf{R}_{i-1}^i \mathbf{I}_i \mathbf{R}_i^{i-1})_{33}$ represents the element of the third column and the third row in matrix $(\mathbf{R}_{i-1}^i \mathbf{I}_i \mathbf{R}_i^{i-1})$.

Again, Differentiating Equation (4.8) gives

$$k_i \dot{\delta}_i = (\mathbf{R}_{i-1}^i \Omega_i \mathbf{n}_i + \mathbf{R}_{i-1}^i \dot{\mathbf{n}}_i) \cdot \mathbf{k} + (\mathbf{R}_{i-1}^i \mathbf{n}_i) \cdot \Omega_{i-1} \mathbf{k} \quad (5.16)$$

Similarly, referring to Equations (4.17), (4.18), (4.21), (4.25) and (4.28), Equation (5.16) becomes

$$\begin{aligned}
\ddot{q}_i = & \frac{1}{(\mathbf{R}_{i-1}^i \mathbf{I}_i \mathbf{R}_i^{i-1})_{33}} \left(k_i \dot{\delta}_i - \left\{ \mathbf{R}_{i-1}^i \boldsymbol{\Omega}_i \mathbf{n}_i + \mathbf{R}_{i-1}^i \left[2\boldsymbol{\Omega}_i \mathbf{I}_i \dot{\omega}_i + \boldsymbol{\Omega}_i \boldsymbol{\Omega}_i \mathbf{I}_i \omega_i \right. \right. \right. \\
& + \dot{\boldsymbol{\Omega}}_i \mathbf{I}_i \omega_i - \boldsymbol{\Omega}_i \mathbf{N}_i + \mathbf{I}_i \mathbf{R}_i^{i-1} \left(\boldsymbol{\Omega}_{i-1} \boldsymbol{\Omega}_{i-1} \omega_{l,i} + \boldsymbol{\Omega}_{i-1} \dot{\omega}_{i-1} + 2\boldsymbol{\Omega}_{i-1} \dot{\omega}_{l,i} \right. \\
& + \dot{\boldsymbol{\Omega}}_{i-1} \omega_{l,i} + \ddot{\omega}_{i-1} \left. \right) + \boldsymbol{\Omega}_i (\mathbf{S} \mathbf{l}_i \times \mathbf{f}_i) + \mathbf{S} \mathbf{l}_i \times \dot{\mathbf{f}}_i + \boldsymbol{\Omega}_i (\mathbf{C} \mathbf{l}_i \times \mathbf{F}_i) \\
& \left. \left. \left. + \mathbf{C} \mathbf{l}_i \times \dot{\mathbf{F}}_i \right] \right\} \cdot \mathbf{k} \right) - \frac{1}{(\mathbf{R}_{i-1}^i \mathbf{I}_i \mathbf{R}_i^{i-1})_{33}} (\mathbf{R}_{i-1}^i \mathbf{n}_i) \cdot \boldsymbol{\Omega}_i \mathbf{k} \quad (4.48)
\end{aligned}$$

where

$$\dot{\delta}_i = G_i \dot{q}_i - \dot{q}_{mi}$$

For the rigid-body controller case, as δ_i approaches zero, the number of degree of freedom of link i reduces to one. The system governing equation then becomes a second-order ordinary differential equation. Since the required feedbacks q_i and $\dot{q}_i$ are measurable, no estimation is needed.

VI. SUMMARY

A. CONCLUSIONS

A direct dynamic model and an inverse dynamic model have been developed for multi-link robot manipulators with joint flexibility. Since an additional degree of freedom was considered for joint flexibility, two sets of system dynamic equations were derived by the Lagrangian method for direct dynamics which returns the robot joint motions according to input torques. The first equation described the large motions of links while the second described the small motions of twist angles of joint springs. These two equations are expressed in a form suited for the sequential integration method [Ref. 8]. Moreover, the Lagrangian method is systematic and easy to program.

The Newton-Euler method was used to develop the inverse dynamics which returns the required torques assuming the robot joint motions are known. The Newton-Euler method provides efficient computation and has potential for the on-line control purpose. The inverse dynamics for multi-link robot manipulators was derived by decoupling the link dynamics and motor dynamics. A fourth-order system equation of link variables was obtained and a fourth-order error dynamics was designed for controller. This control was finally achieved by the design of a recursive estimator which estimates the second and third time derivatives of link positions. The advantage of the estimator is to eliminate the differentiation process that would cause noise problems.

B. RECOMMENDATIONS

The recommendations are listed as follows:

1. Continue to perform computer simulation for multi-link robot manipulators with joint flexibility.
2. Evaluate the performance of flexible-body controller.
3. Implement the flexible-body controller for on-line control.

APPENDIX A. DERIVATION OF THE RESIDUAL TERM

The residual acceleration mentioned in this thesis includes all the nonlinear terms caused by the Coriolis acceleration and centripetal acceleration due to link motion. The residual accelerations can be computed through a recursive form and computational efficiency can be therefore improved since the detailed computational procedure of each nonlinear term is skipped.

Recall Equation (3.26):

$$\ddot{\mathbf{W}}_i = \sum_{k=1}^i \sum_{m=1}^i \mathbf{W}_{i,km} \dot{\mathbf{q}}_m \dot{\mathbf{q}}_k + \sum_{k=1}^i \mathbf{W}_{i,k} \ddot{\mathbf{q}}_k \quad (\text{A.1})$$

The residual acceleration is defined as

$$\mathbf{W}_{ri} = \sum_{k=1}^i \sum_{m=1}^i \mathbf{W}_{i,km} \dot{\mathbf{q}}_m \dot{\mathbf{q}}_k \quad (\text{A.2})$$

The recursive form $\mathbf{W}_{ri}$ is derived as follows. The derivation begins with the recursive relation of homogeneous transformation matrix $\mathbf{W}_i$, $\dot{\mathbf{W}}_i$ and $\ddot{\mathbf{W}}_i$.

$$\mathbf{W}_i = \mathbf{W}_{i-1} \mathbf{A}_i \quad (\text{A.3})$$

Therefore,

$$\dot{\mathbf{W}}_i = \dot{\mathbf{W}}_{i-1} \mathbf{A}_i + \mathbf{W}_{i-1} \dot{\mathbf{A}}_i \quad (\text{A.4})$$

$$\ddot{\mathbf{W}}_i = \ddot{\mathbf{W}}_{i-1} \mathbf{A}_i + 2\dot{\mathbf{W}}_{i-1} \dot{\mathbf{A}}_i + \mathbf{W}_{i-1} \ddot{\mathbf{A}}_i \quad (\text{A.5})$$

since

$$\dot{\mathbf{A}}_i = \mathbf{Q} \mathbf{A}_i \dot{\mathbf{q}}_i \quad (\text{A.6})$$

$$\ddot{\mathbf{A}}_i = \mathbf{Q} \dot{\mathbf{A}}_i \dot{\mathbf{q}}_i + \mathbf{Q} \mathbf{A}_i \ddot{\mathbf{q}}_i \quad (\text{A.7})$$

where

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (A.8)$$

is called the operative matrix for revolute joints. For prismatic joints, it becomes

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (A.9)$$

The recursive form of $\ddot{W}_i$ is rewritten as

$$\ddot{W}_i = \ddot{W}_{i-1}A_i + \dot{W}_{i-1}\dot{A}_i + \dot{W}_{i-1}QA_i\dot{q}_i + W_{i-1}Q\dot{A}_i\dot{q}_i + W_{i-1}QA_i\ddot{q}_i \quad (A.10)$$

If $i=1$,

$$\ddot{W}_1 = \ddot{W}_0A_1 + \dot{W}_0\dot{A}_1 + \dot{W}_0QA_1\dot{q}_1 + W_0Q\dot{A}_1\dot{q}_1 + W_0QA_1\ddot{q}_1 \quad (A.11)$$

where

$$W_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (A.12)$$

then

$$\ddot{W}_0 = \dot{W}_0 = 0 \quad (A.13)$$

and (A.11) becomes

$$\begin{aligned} \ddot{W}_1 &= W_0 Q \dot{A}_1 \dot{q}_1 + W_0 Q A_1 \ddot{q}_1 \\ &= W_0 Q Q A_1 \dot{q}_1 \dot{q}_1 + W_0 Q A_1 \ddot{q}_1 \end{aligned} \quad (A.14)$$

According to Equation (A.1), Equation (A.14) can be expressed as

$$\ddot{W}_1 = W_{r1} + W_0 Q A_1 \ddot{q}_1 \quad (A.15)$$

where

$$W_{r1} = W_0 Q Q A_1 \dot{q}_1 \dot{q}_1 \quad (A.16)$$

If $i=2$,

$$\begin{aligned} \ddot{W}_2 &= \ddot{W}_1 A_2 + \dot{W}_1 \dot{A}_2 + \dot{W}_1 Q A_2 \dot{q}_2 + W_1 Q \dot{A}_2 \dot{q}_2 + W_1 Q A_2 \ddot{q}_2 \\ &= \ddot{W}_1 A_2 + \dot{W}_1 Q A_2 \dot{q}_2 + \dot{W}_1 Q A_2 \dot{q}_2 + W_1 Q Q A_2 \dot{q}_2 \dot{q}_2 + W_1 Q A_2 \ddot{q}_2 \\ &= \ddot{W}_1 A_2 + 2 \dot{W}_1 Q A_2 \dot{q}_2 + W_1 Q Q A_2 \dot{q}_2 \dot{q}_2 + W_1 Q A_2 \ddot{q}_2 \end{aligned} \quad (A.17)$$

Substituting equation (A.15) into (A.17) yields

$$\begin{aligned} \ddot{W}_2 &= (W_{r1} + W_0 Q A_1 \ddot{q}_1) A_2 + 2 \dot{W}_1 Q A_2 \dot{q}_2 + W_1 Q Q A_2 \dot{q}_2 \dot{q}_2 + W_1 Q A_2 \ddot{q}_2 \\ &= (W_{r1} A_2 + 2 \dot{W}_1 Q A_2 \dot{q}_2 + W_1 Q Q A_2 \dot{q}_2 \dot{q}_2) + W_0 Q A_1 A_2 \ddot{q}_1 \\ &\quad + W_1 Q A_2 \ddot{q}_2 \end{aligned} \quad (A.18)$$

where

$$W_{r2} = W_{r1} A_2 + 2 \dot{W}_1 Q A_2 \dot{q}_2 + W_1 Q Q A_2 \dot{q}_2 \dot{q}_2 \quad (A.19)$$

Similarly, the rest of the residual terms can be found in a recursive manner as $i=3,4,5,6$ and so on. Therefore, a general form of residual terms can be obtained as

$$W_{ri} = W_{ri-1} A_i + 2 \dot{W}_{i-1} Q A_i \dot{q}_i + W_{i-1} Q Q A_i \dot{q}_i \dot{q}_i \quad (A.20)$$

where $i=1,2, \dots, 6$

APPENDIX B. DERIVATION OF DIRECT DYNAMICS

A direct dynamics is modeled by using the Lagrangian method. The Lagrangian equation is given as

$$\frac{d}{dt} \left(\frac{\partial(KE)}{\partial \dot{\xi}_j} \right) - \frac{\partial(KE)}{\partial \xi_j} + \frac{\partial(PE)}{\partial \xi_j} = P_j \quad (j=1, 2, \dots, n) \quad (3.1)$$

where ξ are the generalized coordinates, n is the number of degrees of freedom and P_j denotes the generalized forces.

The generalized coordinates are defined as

$$\xi = [q_1, q_2, \dots, q_6, \delta_1, \delta_2, \dots, \delta_6]^T$$

where the generalized coordinate q_i for large motion represents the angular position of link i and the generalized coordinate δ_i for small motion represents the angular displacement between q_i (i.e., link position variable) and motor position variable q_{mi} .

The generalized force P_j obtained from the concept of virtual work is defined as

$$\delta W = \sum_{i=1}^6 T_i \delta q_{mi} = \sum_{i=1}^6 T_i (G_i \delta q_i - \delta \delta_i)$$

where $T_i G_i$ is the generalized force acting on the generalized coordinate q_i , and $-T_i$ is the generalized force acting on the generalized coordinate δ_i .

Two dynamic equations, named the dynamic equations of large motion and the dynamic equations of small motion according to generalized

coordinates, are derived separately in general forms for the sequential integration method in simulation.

A. THE DYNAMIC EQUATION OF LARGE MOTION

The dynamic equation of large motion is derived from Lagrangian equation with generalized coordinate of link. Performing the differentiation of Equation (3.7) with respect to $\dot{q}_j$ obtains

$$\frac{\partial(KE)_{LK}}{\partial \dot{q}_j} = \frac{1}{2} \sum_{i=j}^6 \text{tr} \left[\frac{\partial \dot{\mathbf{W}}_i}{\partial \dot{q}_j} \mathbf{J}_i \mathbf{W}_i^\tau + \dot{\mathbf{W}}_i \mathbf{J}_i \left(\frac{\partial \dot{\mathbf{W}}_i}{\partial \dot{q}_j} \right)^\tau \right] \quad (\text{B.1})$$

by introducing the notation

$$\frac{\partial \dot{\mathbf{W}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{W}_i}{\partial q_j} = \mathbf{W}_{i,j} \quad \text{for } i \geq j \quad (\text{B.2})$$

Equation (B.1) can be rewritten as

$$\frac{\partial(KE)_{LK}}{\partial \dot{q}_j} = \frac{1}{2} \sum_{i=j}^6 \text{tr} \left(\mathbf{W}_{i,j} \mathbf{J}_i \dot{\mathbf{W}}_i^\tau + \dot{\mathbf{W}}_i \mathbf{J}_i \mathbf{W}_{i,j}^\tau \right) \quad (\text{B.3})$$

where

$$\text{tr} \left(\mathbf{W}_{i,j} \mathbf{J}_i \dot{\mathbf{W}}_i^\tau \right) = \text{tr} \left(\dot{\mathbf{W}}_i \mathbf{J}_i \mathbf{W}_{i,j}^\tau \right) \quad (\text{B.4})$$

and (B.3) becomes

$$\frac{\partial(KE)_{LK}}{\partial \dot{q}_j} = \sum_{i=j}^6 \text{tr} \left(\mathbf{W}_{i,j} \mathbf{J}_i \dot{\mathbf{W}}_i^\tau \right) \quad (\text{B.5})$$

by substituting Equation (B.4) into Equation (B.3).

Differentiate Equation (B.5) with respect to time gives

$$\frac{d}{dt} \left[\frac{\partial(\text{KE})_{\text{LK}}}{\partial \dot{q}_j} \right] = \sum_{i=j}^6 \text{tr}(\dot{\mathbf{W}}_{i,j} \mathbf{J}_i \mathbf{W}_i^\top + \mathbf{W}_{i,j} \mathbf{J}_i \ddot{\mathbf{W}}_i^\top) \quad (\text{B.6})$$

since

$$\dot{\mathbf{W}}_{i,j} = \sum_{k=1}^i \mathbf{W}_{i,j,k} \dot{q}_k \quad (\text{B.7})$$

Equation (B.6) becomes

$$\frac{d}{dt} \left[\frac{\partial(\text{KE})_{\text{LK}}}{\partial \dot{q}_j} \right] = \sum_{i=j}^6 \text{tr} \left[\left(\sum_{k=1}^i \mathbf{W}_{i,j,k} \dot{q}_k \right) \mathbf{J}_i \dot{\mathbf{W}}_i^\top + \mathbf{W}_{i,j} \mathbf{J}_i \ddot{\mathbf{W}}_i^\top \right] \quad (\text{B.8})$$

The second term of the Lagrange equation due to link kinetic energy is obtained by differentiating Equation (3.7) with respect to q_j :

$$\begin{aligned} \frac{\partial(\text{KE})_{\text{LK}}}{\partial q_j} &= \frac{1}{2} \sum_{i=j}^6 \text{tr} \left[\frac{\partial \dot{\mathbf{W}}_i}{\partial q_j} \mathbf{J}_i \dot{\mathbf{W}}_i^\top + \dot{\mathbf{W}}_i \mathbf{J}_i \left(\frac{\partial \dot{\mathbf{W}}_i}{\partial q_j} \right)^\top \right] \\ &= \sum_{i=j}^6 \text{tr} \left(\frac{\partial \dot{\mathbf{W}}_i}{\partial q_j} \mathbf{J}_i \dot{\mathbf{W}}_i^\top \right) \end{aligned} \quad (\text{B.9})$$

where

$$\text{tr} \left(\frac{\partial \dot{\mathbf{W}}_i}{\partial q_j} \mathbf{J}_i \dot{\mathbf{W}}_i^\top \right) = \text{tr} \left[\dot{\mathbf{W}}_i \mathbf{J}_i \left(\frac{\partial \dot{\mathbf{W}}_i}{\partial q_j} \right)^\top \right]$$

and

$$\dot{\mathbf{W}}_i = \sum_{k=1}^i \mathbf{W}_{i,k} \dot{q}_k \quad (\text{B.10})$$

$$\frac{\partial \dot{\mathbf{W}}_i}{\partial \dot{\mathbf{q}}_j} = \dot{\mathbf{W}}_{i,j} = \sum_{k=1}^i \mathbf{W}_{i,j,k} \dot{\mathbf{q}}_k \quad \text{for } i \geq j \quad (\text{B.11})$$

Equation (B.9) then becomes

$$\frac{\partial(\text{KE})_{\text{LK}}}{\partial \dot{\mathbf{q}}_j} = \sum_{i=j}^6 \text{tr} \left[\left(\sum_{k=1}^i \mathbf{W}_{i,j,k} \dot{\mathbf{q}}_k \right) \mathbf{J}_i \dot{\mathbf{W}}_i^T \right] \quad (\text{B.12})$$

Similarly, performing the differentiation of Equation (3.14) with respect to $\dot{\mathbf{q}}_j$ obtains

$$\begin{aligned} \frac{\partial(\text{KE})_m}{\partial \dot{\mathbf{q}}_j} = & \sum_{i=1}^6 \frac{1}{2} \text{tr} \left\{ \left[\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\mathbf{q}}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \frac{\partial \mathbf{A}_{mi}}{\partial \dot{\mathbf{q}}_j} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right. \right. \\ & + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \mathbf{A}_{mi}}{\partial \dot{\mathbf{q}}_j} \right)^T \dot{\mathbf{W}}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \left(\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\mathbf{q}}_j} \right)^T \left. \right] \\ & + 2 \left[\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\mathbf{q}}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \frac{\partial \mathbf{A}_{mi}}{\partial \dot{\mathbf{q}}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \\ & + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\mathbf{q}}_j} \right)^T \mathbf{W}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi} \left(\frac{\partial \mathbf{W}_{i-1}}{\partial \dot{\mathbf{q}}_j} \right)^T \left. \right] \\ & + \left[\frac{\partial \mathbf{W}_{i-1}}{\partial \dot{\mathbf{q}}_j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1} \frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\mathbf{q}}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \\ & \left. \left. + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\mathbf{q}}_j} \right)^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \frac{\partial \mathbf{W}_{i-1}^T}{\partial \dot{\mathbf{q}}_j} \right] \right\} \quad (\text{B.13}) \end{aligned}$$

Since $\mathbf{A}_{mi}$ are not functions of $\dot{\mathbf{q}}_j$, and $i \geq j+1$, thus

$$\frac{\partial \mathbf{A}_{mi}}{\partial \dot{\mathbf{q}}_j} = 0$$

$$\left(\frac{\partial \mathbf{W}_{i-1}}{\partial \dot{\mathbf{q}}_j} \right) = 0$$

By introducing the notation

$$\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\mathbf{q}}_j} = \mathbf{W}_{i-1,j} \quad (\text{B.14})$$

Equation (B.13) becomes

$$\begin{aligned} \frac{\partial (\text{KE})_m}{\partial \dot{\mathbf{q}}_j} = & \sum_{i=j+1}^6 \frac{1}{2} \text{tr} \left\{ \left(\mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \mathbf{W}_{i-1,j}^T \right) \right. \\ & + 2 \left[\mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\mathbf{q}}_j} \right)^T \mathbf{W}_{i-1}^T \right] \\ & \left. + \left[\mathbf{W}_{i-1} \frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\mathbf{q}}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\mathbf{q}}_j} \right)^T \mathbf{W}_{i-1}^T \right] \right\} \quad (\text{B.15}) \end{aligned}$$

as

$$\text{tr} \left(\mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right) = \text{tr} \left(\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \mathbf{W}_{i-1,j}^T \right) \quad (\text{B.16})$$

and

$$\text{tr} \left(\mathbf{W}_{i-1} \frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\mathbf{q}}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right) = \text{tr} \left[\mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\mathbf{q}}_j} \right)^T \mathbf{W}_{i-1}^T \right] \quad (\text{B.17})$$

one can finally obtain

$$\begin{aligned}
\frac{\partial(\text{KE})_m}{\partial \dot{q}_j} &= \sum_{i=j+1}^6 \text{tr} \left[\mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^\top \dot{\mathbf{W}}_{i-1}^\top + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^\top \mathbf{W}_{i-1}^\top \right] \\
&+ \sum_{i=1}^6 \text{tr} \left[\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{q}_j} \right)^\top \mathbf{W}_{i-1}^\top + \mathbf{W}_{i-1} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{q}_j} \right) \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^\top \mathbf{W}_{i-1}^\top \right]
\end{aligned} \tag{B.18}$$

Introducing the notations as follows:

$$\dot{\mathbf{A}}_{mi} = \frac{\partial \mathbf{A}_{mi}}{\partial q_{mj}} \dot{q}_{mj} = \mathbf{A}_{mj,j} \quad \text{where } i=j \tag{B.19}$$

$$\dot{q}_{mj} = \mathbf{G}_j \dot{q}_j - \dot{\delta}_j \tag{B.20}$$

$$\sum_{i=1}^6 \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{q}_j} \right) = \begin{cases} \mathbf{A}_{mj,j} \mathbf{G}_j, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases} \tag{B.21}$$

substituting Equation (B.21) into Equation (B.18), Equation (B.18) becomes

$$\begin{aligned}
\frac{\partial(\text{KE})_m}{\partial \dot{q}_j} &= \sum_{i=j+1}^6 \text{tr} \left[\mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^\top \dot{\mathbf{W}}_{i-1}^\top + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^\top \mathbf{W}_{i-1}^\top \right] \\
&+ \text{tr} \left[\dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \mathbf{A}_{mj,j}^\top \mathbf{W}_{j-1}^\top + \mathbf{W}_{j-1} \mathbf{G}_j \mathbf{A}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top \right] \tag{B.22}
\end{aligned}$$

Differentiating Equation (B.22) with respect to time obtains

$$\begin{aligned}
\frac{d}{dt} \left[\frac{\partial(\text{KE})_m}{\partial \dot{q}_j} \right] &= \text{tr} \left(\ddot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \mathbf{A}_{mj,j}^\top \mathbf{W}_{j-1}^\top + \dot{\mathbf{W}}_{j-1} \dot{\mathbf{A}}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \mathbf{A}_{mj,j}^\top \mathbf{W}_{j-1}^\top \right. \\
&+ \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \dot{\mathbf{A}}_{mj,j}^\top \mathbf{W}_{j-1}^\top + \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \mathbf{A}_{mj,j}^\top \dot{\mathbf{W}}_{j-1}^\top \\
&+ \dot{\mathbf{W}}_{j-1} \mathbf{G}_j \mathbf{A}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top + \mathbf{W}_{j-1} \mathbf{G}_j \dot{\mathbf{A}}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top \\
&\left. + \mathbf{W}_{j-1} \mathbf{G}_j \mathbf{A}_{mj,j} \mathbf{J}_{mj} \ddot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top + \mathbf{W}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \dot{\mathbf{A}}_{mj,j}^\top \dot{\mathbf{W}}_{j-1}^\top \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=j+1}^6 \text{tr} \left(\dot{\mathbf{W}}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \mathbf{W}_{i-1,j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right. \\
& + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \ddot{\mathbf{W}}_{i-1}^T \\
& + \dot{\mathbf{W}}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1,j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \\
& \left. + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \ddot{\mathbf{A}}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right) \quad (\text{B.23})
\end{aligned}$$

where

$$\text{tr} \left(\dot{\mathbf{W}}_{j-1} \dot{\mathbf{A}}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \mathbf{A}_{mj,j}^T \mathbf{W}_{j-1}^T \right) = \text{tr} \left(\mathbf{W}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \dot{\mathbf{A}}_{mj,j}^T \dot{\mathbf{W}}_{j-1}^T \right)$$

Equation (B.23) then becomes

$$\begin{aligned}
\frac{d}{dt} \left[\frac{\partial (\text{KE})_m}{\partial \dot{\mathbf{q}}_j} \right] & = \text{tr} \left(\ddot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \mathbf{A}_{mj,j}^T \mathbf{W}_{j-1}^T + 2 \dot{\mathbf{W}}_{j-1} \dot{\mathbf{A}}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \mathbf{A}_{mj,j}^T \mathbf{W}_{j-1}^T \right. \\
& + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \ddot{\mathbf{W}}_{i-1}^T \\
& + \dot{\mathbf{W}}_{j-1} \mathbf{G}_j \mathbf{A}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^T \mathbf{W}_{j-1}^T + \mathbf{W}_{j-1} \mathbf{G}_j \dot{\mathbf{A}}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^T \mathbf{W}_{j-1}^T \\
& + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \ddot{\mathbf{A}}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \left. \right) \\
& + \sum_{i=j+1}^6 \text{tr} \left(\dot{\mathbf{W}}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \mathbf{W}_{i-1,j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right. \\
& + 2 \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \ddot{\mathbf{W}}_{i-1}^T \\
& + \dot{\mathbf{W}}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1,j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \\
& \left. + \mathbf{W}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \ddot{\mathbf{A}}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right) \quad (\text{B.24})
\end{aligned}$$

The second term of Equation (3.1) due to motor kinetic energy is

$$\begin{aligned}
\frac{\partial (\text{KE})_m}{\partial \mathbf{q}_j} & = \sum_{i=1}^6 \frac{1}{2} \text{tr} \left\{ \left[\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \mathbf{q}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \frac{\partial \mathbf{A}_{mi}}{\partial \mathbf{q}_j} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right. \right. \\
& \left. \left. + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \mathbf{A}_{mi}}{\partial \mathbf{q}_j} \right)^T \dot{\mathbf{W}}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \left(\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \mathbf{q}_j} \right)^T \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& +2 \left[\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \mathbf{q}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \frac{\partial \mathbf{A}_{mi}}{\partial \mathbf{q}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \\
& \left. + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \mathbf{q}_j} \mathbf{W}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \left(\frac{\partial \mathbf{W}_{i-1}}{\partial \mathbf{q}_j} \right)^T \right] \\
& + \left[\frac{\partial \mathbf{W}_{i-1}}{\partial \mathbf{q}_j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1} \frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \mathbf{q}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \\
& \left. + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \mathbf{q}_j} \right)^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \left(\frac{\partial \mathbf{W}_{i-1}}{\partial \mathbf{q}_j} \right)^T \right] \Bigg\} \quad (\text{B.25})
\end{aligned}$$

As the terms $\frac{\partial \mathbf{W}_{i-1}}{\partial \mathbf{q}_j}$ and $\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \mathbf{q}_j}$ exist only when $i \geq j+1$, the terms

$\frac{\partial \mathbf{A}_{mi}}{\partial \mathbf{q}_j}$ and $\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \mathbf{q}_j}$ exist only when $i \geq j$, and

$$\begin{aligned}
\text{tr} \left(\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \mathbf{q}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right) &= \text{tr} \left[\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \left(\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \mathbf{q}_j} \right)^T \right] \\
\text{tr} \left(\dot{\mathbf{W}}_{i-1} \frac{\partial \mathbf{A}_{mi}}{\partial \mathbf{q}_j} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right) &= \text{tr} \left[\dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \mathbf{A}_{mi}}{\partial \mathbf{q}_j} \right)^T \dot{\mathbf{W}}_{i-1}^T \right] \\
\text{tr} \left(\frac{\partial \mathbf{W}_{i-1}}{\partial \mathbf{q}_j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right) &= \text{tr} \left[\mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \left(\frac{\partial \mathbf{W}_{i-1}}{\partial \mathbf{q}_j} \right)^T \right]
\end{aligned}$$

$$\text{tr} \left(\mathbf{W}_{i-1} \frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \mathbf{q}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right) = \text{tr} \left(\mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \mathbf{q}_j} \right)^T \mathbf{W}_{i-1}^T \right)$$

Equation (B.25) can be rewritten as

$$\begin{aligned} \frac{\partial (\text{KE})_m}{\partial \mathbf{q}_j} = & \sum_{i=j+1}^6 \frac{1}{2} \text{tr} \left\{ 2 \frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \mathbf{q}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + 2 \frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \mathbf{q}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \\ & \left. + 2 \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \left(\frac{\partial \mathbf{W}_{i-1}}{\partial \mathbf{q}_j} \right)^T + 2 \frac{\partial \mathbf{W}_{i-1}}{\partial \mathbf{q}_j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right\} \\ & + \frac{1}{2} \text{tr} \left[2 \dot{\mathbf{W}}_{j-1} \frac{\partial \mathbf{A}_{mj}}{\partial \mathbf{q}_j} \mathbf{J}_{mj} \mathbf{A}_{mj}^T \dot{\mathbf{W}}_{j-1}^T + 2 \dot{\mathbf{W}}_{j-1} \frac{\partial \mathbf{A}_{mj}}{\partial \mathbf{q}_j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^T \mathbf{W}_{j-1}^T \right. \\ & \left. + 2 \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \left(\frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \mathbf{q}_j} \right)^T \dot{\mathbf{W}}_{j-1}^T + 2 \mathbf{W}_{j-1} \frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \mathbf{q}_j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^T \mathbf{W}_{j-1}^T \right] \end{aligned} \quad (\text{B.26})$$

Introducing the notation

$$\frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \mathbf{q}_j} = \dot{\mathbf{A}}_{mj,j} \quad (\text{B.27})$$

Equation (B.26) becomes

$$\begin{aligned} \frac{\partial (\text{KE})_m}{\partial \mathbf{q}_j} = & \sum_{i=j+1}^6 \text{tr} \left(\dot{\mathbf{W}}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \dot{\mathbf{W}}_{i-1,j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right. \\ & \left. + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1,j}^T + \mathbf{W}_{i-1,j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right) \\ & + \text{tr} \left(\dot{\mathbf{W}}_{j-1} \mathbf{G}_j \mathbf{A}_{mj,j} \mathbf{J}_{mj} \mathbf{A}_{mj}^T \dot{\mathbf{W}}_{j-1}^T + \dot{\mathbf{W}}_{j-1} \mathbf{G}_j \mathbf{A}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^T \mathbf{W}_{j-1}^T \right. \\ & \left. + \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{G}_j \dot{\mathbf{A}}_{mj,j}^T \mathbf{W}_{j-1}^T + \mathbf{W}_{j-1} \mathbf{G}_j \dot{\mathbf{A}}_{mj,j}^T \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^T \mathbf{W}_{j-1}^T \right) \end{aligned} \quad (\text{B.28})$$

The last term of Equation (3.1) is obtained by differentiating Equation (3.20)

$$\frac{\partial (\text{PE})}{\partial q_j} = -\sum_{i=j}^6 m_i \mathbf{r}_i^{i^T} \mathbf{W}_{i,j}^T \mathbf{g} - m_{mj} \mathbf{r}_{mj}^{j^T} \mathbf{A}_{mj,j}^T \mathbf{W}_{j-1}^T \mathbf{g} - \sum_{i=j+1}^6 m_{mi} \mathbf{r}_{mi}^{i^T} \mathbf{A}_{mi}^T \mathbf{W}_{i-1,j}^T \mathbf{g} \quad (\text{B.29})$$

Introducing the notations

$$\mathbf{X}_j = \mathbf{W}_{j-1} \mathbf{A}_{mj} \quad (\text{B.30})$$

$$\dot{\mathbf{X}}_j = \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} + \mathbf{W}_{j-1} \dot{\mathbf{A}}_{mj} \quad (\text{B.31})$$

$$\ddot{\mathbf{X}}_j = \ddot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} + 2 \dot{\mathbf{W}}_{j-1} \dot{\mathbf{A}}_{mj} \mathbf{W}_{j-1} \ddot{\mathbf{A}}_{mj} \quad (\text{B.32})$$

and combining Equations (B.8), (B.12), (B.24), (B.28) and (B.29) according to Equation (3.1) gives the final general form of dynamic equation of large motion, i.e.,

$$\begin{aligned} & \sum_{i=j}^6 \text{tr}(\mathbf{W}_{i,j} \mathbf{J}_i \ddot{\mathbf{W}}_i^T) + \text{tr}(\ddot{\mathbf{X}}_j \mathbf{J}_{mj} \mathbf{A}_{mj,j}^T \mathbf{W}_{j-1}^T) + \sum_{i=j+1}^6 \text{tr}(\ddot{\mathbf{X}}_j \mathbf{J}_{mj} \mathbf{A}_{mj}^T \mathbf{W}_{i-1,j}^T) \\ & - \sum_{i=j}^6 m_i \mathbf{r}_i^{i^T} \mathbf{W}_{i,j}^T \mathbf{g} - m_{mj} \mathbf{r}_{mj}^{j^T} \mathbf{A}_{mj,j}^T \mathbf{W}_{j-1}^T \mathbf{g} - \sum_{i=j+1}^6 m_{mi} \mathbf{r}_{mi}^{i^T} \mathbf{A}_{mi}^T \mathbf{W}_{i-1,j}^T \mathbf{g} = \mathbf{G}_j \mathbf{T}_j \end{aligned} \quad (\text{B.33})$$

B. THE DYNAMIC EQUATION OF SMALL MOTION

The dynamic equation of small motion is derived from Lagrangian equation with generalized coordinate of deflection. The first term in Equation(3.1) can be written as

$$\frac{d}{dt} \left(\frac{\partial (\text{KE})}{\partial \dot{\delta}_j} \right) = \frac{d}{dt} \left(\frac{\partial (\text{KE})_{\text{LK}}}{\partial \dot{\delta}_j} \right) + \frac{d}{dt} \left(\frac{\partial (\text{KE})_{\text{m}}}{\partial \dot{\delta}_j} \right)$$

where

$$\frac{d}{dt} \left(\frac{\partial (\text{KE})_{\text{LK}}}{\partial \dot{\delta}_j} \right) = 0$$

therefore, performing the differentiation of Equation (3.14) with respect to $\dot{\delta}_j$ gives

$$\begin{aligned} \frac{\partial (\text{KE})_{\text{m}}}{\partial \dot{\delta}_j} = & \sum_{i=1}^6 \frac{1}{2} \text{tr} \left\{ \left[\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\delta}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \frac{\partial \mathbf{A}_{mi}}{\partial \dot{\delta}_j} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \dot{\mathbf{W}}_{i-1}^T \right. \right. \\ & \left. \left. + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \mathbf{A}_{mi}}{\partial \dot{\delta}_j} \right)^T \dot{\mathbf{W}}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \mathbf{A}_{mi}^T \left(\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\delta}_j} \right)^T \right] \right. \\ & + 2 \left[\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\delta}_j} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \frac{\partial \mathbf{A}_{mi}}{\partial \dot{\delta}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \\ & \left. \left. + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\delta}_j} \right)^T \mathbf{W}_{i-1}^T + \dot{\mathbf{W}}_{i-1} \mathbf{A}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \left(\frac{\partial \mathbf{W}_{i-1}}{\partial \dot{\delta}_j} \right)^T \right] \right. \\ & \left. + \left[\frac{\partial \mathbf{W}_{i-1}}{\partial \dot{\delta}_j} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1} \frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\delta}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right. \right. \\ & \left. \left. + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\delta}_j} \right)^T \mathbf{W}_{i-1}^T + \mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \left(\frac{\partial \mathbf{W}_{i-1}}{\partial \dot{\delta}_j} \right)^T \right] \right\} \quad (\text{B.34}) \end{aligned}$$

As the terms $\frac{\partial \mathbf{W}_{i-1}}{\partial \dot{\delta}_j}$, $\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\delta}_j}$ and $\frac{\partial \mathbf{A}_{mi}}{\partial \dot{\delta}_j}$ are not functions of $\dot{\delta}_j$, thus

$$\frac{\partial \dot{\mathbf{W}}_{i-1}}{\partial \dot{\delta}_j} = 0$$

$$\frac{\partial \mathbf{W}_{i-1}}{\partial \dot{\delta}_j} = 0$$

$$\frac{\partial \mathbf{A}_{mi}}{\partial \dot{\delta}_j} = 0$$

and

$$\text{tr} \left[\mathbf{W}_{i-1} \dot{\mathbf{A}}_{mi} \mathbf{J}_{mi} \left(\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\delta}_j} \right)^T \mathbf{W}_{i-1}^T \right] = \text{tr} \left(\mathbf{W}_{i-1} \frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\delta}_j} \mathbf{J}_{mi} \dot{\mathbf{A}}_{mi}^T \mathbf{W}_{i-1}^T \right)$$

where $\frac{\partial \dot{\mathbf{A}}_{mi}}{\partial \dot{\delta}_j}$ exists only when $i \geq j$, Equation (B.34) then becomes

$$\frac{\partial (\text{KE})_m}{\partial \dot{\delta}_j} = \frac{1}{2} \text{tr} \left[2 \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \left(\frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \dot{\delta}_j} \right)^T \mathbf{W}_{j-1}^T + 2 \mathbf{W}_{j-1} \left(\frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \dot{\delta}_j} \right)^T \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^T \mathbf{W}_{j-1}^T \right] \quad (\text{B.35})$$

introducing the notation

$$\frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \dot{\delta}_j} = -\mathbf{A}_{mj,j} \quad (\text{B.36})$$

Equation (B.35) becomes

$$\frac{\partial (\text{KE})_m}{\partial \dot{\delta}_j} = \text{tr} \left(-\dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{A}_{mj,j} \mathbf{W}_{j-1}^T - \mathbf{W}_{j-1} \mathbf{A}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^T \mathbf{W}_{j-1}^T \right) \quad (\text{B.37})$$

Differentiating Equation (B.37) with respect to time obtains

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial (\text{KE})_m}{\partial \dot{\delta}_j} \right) = & \text{tr} \left(-\ddot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{A}_{mj,j}^\top \mathbf{W}_{j-1}^\top - \dot{\mathbf{W}}_{j-1} \dot{\mathbf{A}}_{mj} \mathbf{J}_{mj} \mathbf{A}_{mj,j}^\top \mathbf{W}_{j-1}^\top \right. \\
& - \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj,j}^\top \mathbf{W}_{j-1}^\top - \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \mathbf{A}_{mj,j}^\top \dot{\mathbf{W}}_{j-1}^\top \\
& - \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top - \mathbf{W}_{j-1} \dot{\mathbf{A}}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top \\
& \left. - \mathbf{W}_{j-1} \mathbf{A}_{mj,j} \mathbf{J}_{mj} \ddot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top - \mathbf{W}_{j-1} \mathbf{A}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \dot{\mathbf{W}}_{j-1}^\top \right) \quad (\text{B.38})
\end{aligned}$$

The second term of Equation (3.1) is obtained as

$$\begin{aligned}
\frac{\partial (\text{KE})_m}{\partial \delta_j} = & \frac{1}{2} \text{tr} \left[2 \dot{\mathbf{W}}_{j-1} \frac{\partial \mathbf{A}_{mj}}{\partial \delta_j} \mathbf{J}_{mj} \mathbf{A}_{mj}^\top \mathbf{W}_{j-1}^\top + 2 \dot{\mathbf{W}}_{j-1} \frac{\partial \mathbf{A}_{mj}}{\partial \delta_j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top \right. \\
& \left. + 2 \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \left(\frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \delta_j} \right)^\top \mathbf{W}_{j-1}^\top + 2 \mathbf{W}_{j-1} \frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \delta_j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top \right] \quad (\text{B.39})
\end{aligned}$$

and as

$$\frac{\partial \mathbf{A}_{mj}}{\partial \delta_j} = -\mathbf{A}_{mj,j} = \frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \dot{\delta}_j} \quad (\text{B.40})$$

$$\frac{d}{dt} (-\mathbf{A}_{mj,j}) = -\dot{\mathbf{A}}_{mj,j} = \frac{\partial \dot{\mathbf{A}}_{mj}}{\partial \delta_j} \quad (\text{B.41})$$

Equation (B.39) can be rewritten as

$$\begin{aligned}
\frac{\partial (\text{KE})_m}{\partial \delta_j} = & \text{tr} \left[-\dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj,j} \mathbf{J}_{mj} \mathbf{A}_{mj}^\top \mathbf{W}_{j-1}^\top - \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top \right. \\
& \left. - \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj,j}^\top \mathbf{W}_{j-1}^\top - \mathbf{W}_{j-1} \dot{\mathbf{A}}_{mj,j} \mathbf{J}_{mj} \dot{\mathbf{A}}_{mj}^\top \mathbf{W}_{j-1}^\top \right] \quad (\text{B.42})
\end{aligned}$$

The last term of Equation (3.1) is obtained by differentiating Equation (3.20) with respect to δ_j :

$$\frac{\partial (\text{PE})}{\partial \delta_j} = -m_{mj} \mathbf{r}_{mj}^{j^T} \left(-\dot{\mathbf{A}}_{mj,j}^T \right) \mathbf{W}_{j-1}^T \mathbf{g} + k_j \delta_j \quad (\text{B.43})$$

Since

$$\frac{d}{dt} \left(\frac{\partial (\text{KE})_{\text{LK}}}{\partial \dot{\delta}_j} \right) = 0$$

$$\frac{\partial (\text{KE})_{\text{LK}}}{\partial \delta_j} = 0$$

$$\frac{\partial (\text{PE})_{\text{LK}}}{\partial \delta_j} = 0$$

combining Equations (B.38), (B.42) and (B.43) according to Equation (3.1) gives the final general form of dynamic equation of small motion, i.e.,

$$-\text{tr} \left(\ddot{\mathbf{X}}_j \mathbf{J}_{mj} \mathbf{A}_{mj,j}^T \mathbf{W}_{j-1}^T \right) + m_{mj} \mathbf{r}_{mj}^{j^T} \dot{\mathbf{A}}_{mj,j}^T \mathbf{W}_{j-1}^T \mathbf{g} + k_j \delta_j = -T_j \quad (\text{B.44})$$

where

$$\mathbf{X}_j = \mathbf{W}_{j-1} \mathbf{A}_{mj} \quad (\text{B.45})$$

$$\dot{\mathbf{X}}_j = \dot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} + \mathbf{W}_{j-1} \dot{\mathbf{A}}_{mj} \quad (\text{B.46})$$

$$\ddot{\mathbf{X}}_j = \ddot{\mathbf{W}}_{j-1} \mathbf{A}_{mj} + 2\dot{\mathbf{W}}_{j-1} \dot{\mathbf{A}}_{mj} \mathbf{W}_{j-1} \ddot{\mathbf{A}}_{mj} \quad (\text{B.47})$$

APPENDIX C. NEWTON-EULER RECURSIVE FORMULATION

Referring back to the definition of the absolute angular velocity of link i for revolute joint:

$$\omega_i = \mathbf{R}_i^{i-1} (\omega_{i-1} + \omega_{l,i}) \quad (\text{C.1})$$

By premultiplying rotation transformation matrices, the angular velocity of link i is transformed as

$$\mathbf{R}_0^i \omega_i = \mathbf{R}_0^{i-1} (\omega_{i-1} + \omega_{l,i}) \quad (\text{C.2})$$

Differentiating Equation (C.2) w.r.t. time according to $\mathbf{R}_0^i = \mathbf{R}_0^i \Omega_i$ yields

$$\mathbf{R}_0^i \Omega_i \omega_i + \mathbf{R}_0^i \dot{\omega}_i = \mathbf{R}_0^{i-1} \Omega_{i-1} (\omega_{i-1} + \omega_{l,i}) + \mathbf{R}_0^{i-1} (\dot{\omega}_{i-1} + \dot{\omega}_{l,i}) \quad (\text{C.3})$$

since

$$\Omega_i \omega_i = 0$$

equation (C.3) becomes

$$\mathbf{R}_0^i \dot{\omega}_i = \mathbf{R}_0^{i-1} \Omega_{i-1} (\omega_{i-1} + \omega_{l,i}) + \mathbf{R}_0^{i-1} (\dot{\omega}_{i-1} + \dot{\omega}_{l,i}) \quad (\text{C.4})$$

multiplying Equation (C.4) by $\mathbf{R}_i^0$ in both sides gives

$$\dot{\omega}_i = \mathbf{R}_i^{i-1} (\Omega_{i-1} \omega_{l,i} + \dot{\omega}_{i-1} + \dot{\omega}_{l,i}) \quad (\text{C.5})$$

Where

$$\mathbf{R}_0^i \mathbf{R}_i^{i-1} = \mathbf{R}_0^{i-1} \quad \text{and} \quad \mathbf{R}_i^0 \mathbf{R}_0^i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{C.6})$$

Similarly, differentiating Equation (C.6) once and twice w.r.t time gives

$$\ddot{\omega}_i = \mathbf{R}_i^{i-1} \left[\Omega_{i-1} (\Omega_{i-1} \omega_{l,i} + \dot{\omega}_{i-1}) + \dot{\Omega}_{i-1} \omega_{l,i} + 2\Omega_{i-1} \dot{\omega}_{l,i} + \ddot{\omega}_{i-1} + \ddot{\omega}_{l,i} \right] - \Omega_i \dot{\omega}_i \quad (\text{C.7})$$

$$\begin{aligned} \ddot{\omega}_i = \mathbf{R}_i^{i-1} & \left(\Omega_{i-1}^3 \omega_{l,i} + \Omega_{i-1} \Omega_{i-1} \dot{\omega}_{i-1} + 2\Omega_{i-1} \dot{\Omega}_{i-1} \omega_{l,i} + 3\Omega_{i-1} \Omega_{i-1} \dot{\omega}_{l,i} \right. \\ & + 2\Omega_{i-1} \ddot{\omega}_{i-1} + 3\Omega_{i-1} \ddot{\omega}_{l,i} + \dot{\Omega}_{i-1} \Omega_{i-1} \omega_{l,i} + 3\dot{\Omega}_{i-1} \dot{\omega}_{l,i} + \ddot{\Omega}_{i-1} \omega_{l,i} \\ & \left. + \ddot{\omega}_{i-1} + \ddot{\omega}_{l,i} \right) - (\Omega_i \Omega_i \dot{\omega}_i + 2\Omega_i \ddot{\omega}_i) \end{aligned} \quad (\text{C.8})$$

The linear acceleration of link i expressed in terms of base coordinate is given as

$$\mathbf{R}_0^i \mathbf{a}_i = \mathbf{R}_0^i (\mathbf{R}_i^{i-1} \mathbf{a}_{i-1} + \Omega_i \Omega_i \mathbf{S} \mathbf{l}_i + \dot{\Omega}_i \mathbf{S} \mathbf{l}_i) \quad (\text{C.9})$$

differentiating Equation (C.9) w.r.t. time yields

$$\mathbf{R}_0^i \Omega_i \mathbf{a}_i + \mathbf{R}_0^i \dot{\mathbf{a}}_i = \mathbf{R}_0^{i-1} (\Omega_{i-1} \mathbf{a}_{i-1} + \dot{\mathbf{a}}_{i-1}) + \mathbf{R}_0^i (\Omega_i \Omega_i \Omega_i + 2\Omega_i \dot{\Omega}_i + \dot{\Omega}_i \Omega_i + \ddot{\Omega}_i) \mathbf{S} \mathbf{l}_i \quad (\text{C.10})$$

By multiplying $\mathbf{R}_i^0$ in both sides, Equation (C.10) can be rewritten in a recursive form of derivative of the linear acceleration, i.e.,

$$\dot{\mathbf{a}}_i = \mathbf{R}_i^{i-1} (\Omega_{i-1} \mathbf{a}_{i-1} + \dot{\mathbf{a}}_{i-1}) + (\Omega_i^3 + 2\Omega_i \dot{\Omega}_i + \dot{\Omega}_i \Omega_i + \ddot{\Omega}_i) \mathbf{S} \mathbf{l}_i - \Omega_i \mathbf{a}_i \quad (\text{C.11})$$

Similarly, differentiating Equation (C.11) gives the second derivative of linear acceleration, i.e.,

$$\begin{aligned} \ddot{\mathbf{a}}_i = \mathbf{R}_i^{i-1} & \left(\Omega_{i-1}^2 \mathbf{a}_{i-1} + 2\Omega_{i-1} \dot{\mathbf{a}}_{i-1} + \dot{\Omega}_{i-1} \mathbf{a}_{i-1} + \ddot{\mathbf{a}}_{i-1} \right) \\ & + (\Omega_i^4 + 3\Omega_i^2 \dot{\Omega}_i + \dot{\Omega}_i \Omega_i \Omega_i + 2\Omega_i \dot{\Omega}_i \Omega_i + 3\dot{\Omega}_i^2 + 3\Omega_i \ddot{\Omega}_i + \ddot{\Omega}_i \Omega_i + \ddot{\Omega}_i) \mathbf{S} \mathbf{l}_i \\ & - (\Omega_i^2 \mathbf{a}_i + \dot{\Omega}_i \mathbf{a}_i + 2\Omega_i \dot{\mathbf{a}}_i) \end{aligned} \quad (\text{C.12})$$

The derivatives of force and torque can be obtained by following the same way. The results are as follows:

$$\mathbf{f}_i = \mathbf{F}_i + \mathbf{R}_i^{i+1} \mathbf{f}_{i+1} \quad (\text{C.13})$$

$$\dot{\mathbf{f}}_i = \Omega_i \mathbf{F}_i + \dot{\mathbf{F}}_i + \mathbf{R}_i^{i+1} (\Omega_{i+1} \mathbf{f}_{i+1} + \dot{\mathbf{f}}_{i+1}) - \Omega_i \mathbf{f}_i \quad (\text{C.14})$$

$$\begin{aligned} \ddot{\mathbf{f}}_i = & \Omega_i \Omega_i \mathbf{F}_i + 2\Omega_i \dot{\mathbf{F}}_i + \dot{\Omega}_i \mathbf{F}_i + \ddot{\mathbf{F}}_i + \mathbf{R}_i^{i+1} (\Omega_{i+1} \Omega_{i+1} \mathbf{f}_{i+1} + 2\Omega_{i+1} \dot{\mathbf{f}}_{i+1} + \dot{\Omega}_{i+1} \mathbf{f}_{i+1} + \ddot{\mathbf{f}}_{i+1}) \\ & - (\Omega_i \Omega_i \mathbf{f}_i + 2\Omega_i \dot{\mathbf{f}}_i + \dot{\Omega}_i \mathbf{f}_i) \end{aligned} \quad (\text{C.15})$$

$$\mathbf{F}_i = \mathbf{m}_i \mathbf{a}_{ci} \quad (\text{C.16})$$

$$\dot{\mathbf{F}}_i = \mathbf{m}_i \dot{\mathbf{a}}_{ci} \quad (\text{C.17})$$

$$\ddot{\mathbf{F}}_i = \mathbf{m}_i \ddot{\mathbf{a}}_{ci} \quad (\text{C.18})$$

$$\mathbf{a}_{ci} = \mathbf{a}_i + \dot{\Omega}_i \mathbf{Cl}_i + \Omega_i \Omega_i \mathbf{Cl}_i \quad (\text{C.19})$$

$$\dot{\mathbf{a}}_{ci} = \dot{\mathbf{a}}_i + \ddot{\Omega}_i \mathbf{Cl}_i + \dot{\Omega}_i \Omega_i \mathbf{Cl}_i + \Omega_i \dot{\Omega}_i \mathbf{Cl}_i \quad (\text{C.20})$$

$$\ddot{\mathbf{a}}_{ci} = \ddot{\mathbf{a}}_i + \ddot{\Omega}_i \mathbf{Cl}_i + \ddot{\Omega}_i \Omega_i \mathbf{Cl}_i + 2\dot{\Omega}_i \dot{\Omega}_i \mathbf{Cl}_i + \Omega_i \ddot{\Omega}_i \mathbf{Cl}_i \quad (\text{C.21})$$

$$\mathbf{N}_i = \mathbf{I}_i \dot{\omega}_i + \Omega_i \mathbf{I}_i \omega_i \quad (\text{C.22})$$

$$\dot{\mathbf{N}}_i = \mathbf{I}_i \ddot{\omega}_i + \dot{\Omega}_i \mathbf{I}_i \omega_i + \Omega_i \mathbf{I}_i \dot{\omega}_i \quad (\text{C.23})$$

$$\ddot{\mathbf{N}}_i = \mathbf{I}_i \ddot{\omega}_i + \ddot{\Omega}_i \mathbf{I}_i \omega_i + 2\dot{\Omega}_i \mathbf{I}_i \dot{\omega}_i + \Omega_i \mathbf{I}_i \ddot{\omega}_i \quad (\text{C.24})$$

$$\mathbf{n}_i = \mathbf{N}_i + \mathbf{R}_i^{i+1} \mathbf{n}_{i+1} + \mathbf{Sl}_i \times \mathbf{f}_i + \mathbf{Cl}_i \times \mathbf{F}_i \quad (\text{C.25})$$

$$\begin{aligned} \dot{\mathbf{n}}_i = & \Omega_i \mathbf{N}_i + \dot{\mathbf{N}}_i + \mathbf{R}_i^{i+1} (\Omega_{i+1} \mathbf{n}_{i+1} + \dot{\mathbf{n}}_{i+1}) + \Omega_i (\mathbf{Sl}_i \times \mathbf{f}_i) + \mathbf{Sl}_i \times \dot{\mathbf{f}}_i + \Omega_i (\mathbf{Cl}_i \times \mathbf{F}_i) \\ & + \mathbf{Cl}_i \times \dot{\mathbf{F}}_i - \Omega_i \mathbf{n}_i \end{aligned} \quad (\text{C.26})$$

$$\begin{aligned} \ddot{\mathbf{n}}_i = & \Omega_i \Omega_i \mathbf{N}_i + 2\Omega_i \dot{\mathbf{N}}_i + \dot{\Omega}_i \mathbf{N}_i + \ddot{\mathbf{N}}_i + \mathbf{R}_i^{i+1} (\Omega_{i+1} \Omega_{i+1} \mathbf{n}_{i+1} + 2\Omega_{i+1} \dot{\mathbf{n}}_{i+1} \\ & + \dot{\Omega}_{i+1} \mathbf{n}_{i+1} + \ddot{\mathbf{n}}_{i+1}) + \Omega_i \Omega_i (\mathbf{Sl}_i \times \mathbf{f}_i) + 2\Omega_i (\mathbf{Sl}_i \times \dot{\mathbf{f}}_i) + \dot{\Omega}_i (\mathbf{Sl}_i \times \mathbf{f}_i) \\ & + \mathbf{Sl}_i \times \ddot{\mathbf{f}}_i + \Omega_i \Omega_i (\mathbf{Cl}_i \times \mathbf{F}_i) + 2\Omega_i (\mathbf{Cl}_i \times \dot{\mathbf{F}}_i) + \dot{\Omega}_i (\mathbf{Cl}_i \times \mathbf{F}_i) + \mathbf{Cl}_i \times \ddot{\mathbf{F}}_i \\ & - \Omega_i \Omega_i \mathbf{n}_i - 2\Omega_i \dot{\mathbf{n}}_i - \dot{\Omega}_i \mathbf{n}_i \end{aligned} \quad (\text{C.27})$$

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